DOCUMENT RESUME

ED 184 834	SE 030 448
AUTHOR	Allen, Frank B.: And Others
TITLE	Mathematics for High School, Introduction to Matrix Algebra. Preliminary Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	60
NOTE	262p.
EDRS PRICE	MF01/PC11 Plus Postage.
DESCRIPTORS	*Algebra: Experimental Programs: Mathematics Curriculum: *Mathematics Instruction: *Matrices: Modern Mathematics: Secondary Education: *Secondary School Mathematics: Student Research: *Textbooks
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This volume is an experimental edition for a high school course in the theory of matrices and vectors. One of the basic aims is to demonstrate the structure of mathematics. Another criterion is to provide some tools that will be useful in the student's transition from school to college. A last objective is that the intellectually vigorous students may obtain an idea of what constitutes "mathematical research." The five chapters found in the text are: (1) Matrix Operations: (2) The Algebra of 2x2 Matrices: (3) Matrices and Systems of Linear Equations: (4) The Representation of Column Matrices as Geometric Vectors: and (5) Transformations of the Plane. Research Exercises are also included. (Author/MK)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED184834

SCHOOL
MATHEMATICS
STUDY GROUP

MATHEMATICS FOR HIGH SCHOOL

INTRODUCTION TO MATRIX ALGEBRA

(preliminary edition)

U S DEPARTMENT OF HEALTH.
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

"PERMISSION TO REPRODUCE THIS
MATERIAL HAS BEEN GRANTED BY
Mary L. Charles
NSF

TO THE EDUCATIONAL RESOURCES
INFORMATION CENTER (ERIC)."

SE 030 448

MATHEMATICS FOR HIGH SCHOOL

INTRODUCTION TO MATRIX ALGEBRA

(preliminary edition)

Prepared under the supervision of the Panel on Sample Textbooks of the School Mathematics Study Group:

Frank B. Allen, Lyons Township High School

Edwin C. Douglas, Taft School

Donald E. Richmond, Williams College

Charles E. Rickart, Yale University

Henry Swain, New Trier Township High School

Robert J. Walker, Cornell University

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Copyright 1960 by Yale University.

PHOTOLITHOPRINTED BY CUSHING - MALLOY, INC.
ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group. This Study Group includes college and university mathematicians, high school teachers of mathematics, experts in education and representatives of science and technology. The general objective of the Study Group is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisities for the improvement of the teaching of mathematics in our schools is an improved curriculum—one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by the School Mathematics Study Group was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of high school textbooks which would illustrate such an improved curriculum. This textbook is the first product of this project.

The professional mathematicians in the Study Group believe that much of the mathematics presented in this series of texts is important for all well-educated citizens in our society to know and that all of it is important for the pre—college student to learn in preparation for advanced work in the field. At the same time, the high school teachers in the Study Group believe that it is presented in such a form that it can be readily grasped by college capable students.

In most instances the material presented in this series will have a familiar note to it, but the flavor of presentation, the point of view, as it were, will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead a college—bound student to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for later courses.

It is not intended that these books be regarded as the only definitive way of presenting good mathematics to college capable students. Instead, they should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of the commercial textbooks of the future. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

PREFACE

The present volume is an experimental edition for a high-school course in the theory of matrices and vectors. In selecting material for the text, the School Mathematics Study Group has been mindful of the fact that this is the last mathematics course in secondary school, the terminal course for many students. As citizens, they should have a sound idea of the nature of mathematics. This point of view has been emphasized in the Harvard report, "General Education in a Free Society," Harvard University Press, Cambridge, 1945, which states: "Mathematics may be defined as the science of abstract form. The discernment of structure is essential, no less to the appreciation of a painting or symphony than in the behaviour of a physical system; no less in economics than in astronomy. Mathematics studies order, abstracted from the particular objects and phenomena which exhibit it, and in a generalized form."

One of our basic aims is thus to demonstrate the structure of mathematics. We shall not be concerned, however, with structure merely as such. Rather, we shall exhibit some rich mathematics that is totally new to the student and demonstrate structure as we proceed. To make abstract form a topic unto itself often leads to a barren presentation; to discuss the structure of the already-familiar arithmetic and algebra seems forced and repetitive to the boy or girl who is dreaming of a place in a jet age, even in a space age.

It is important to give the student some "new" mathematics that has considerable vigor and vitality. Until very recently, the high-school curriculum has been almost entirely concerned with ideas that were developed during or before the sixteenth and seventeenth centuries. Computers and electronic brains are front-page news. In order to appeal to the imagination of the student and to expose some mathematics that is very much alive, the material must be new, different, and bold.

Another criterion is to provide some tools that will be eminently useful in the student's transition from school to college, tools that will help bridge the gap from the manipulative spirit of high-school mathematics to the abstract viewpoint of modern algebraic studies. Yet this material must not come from the usual sequential courses.

A unit on matrix algebra will satisfy the foregoing criteria. As one operation after another is defined, the structure of mathematics can be repeatedly emphasized. Terms like group, ring, field, and isomorphism will be introduced when meaningful and needed for unifying concepts. Thus they will be met in a new, appropriate, and substantial context; they will not be applied to shopworn material. Introduced by Cayley in 1858, recognized by Heisenberg in 1925 as exactly the tool he needed to develop his revolutionary work in quantum mechanics, employed today in such diverse ways as providing a language for atomic physics, measuring the air flow over the wing of an airplane, and keeping the parts inventory at a minimum in a factory, matrices can put the student close to the frontiers of mathematics and provide striking examples of patterns that arise in the most varied circumstances. Moreover, the student meets some mathematics emancipated from the familiar rules of arithmetic, and he learns that it is within his capacities to "invent" some of his own. If this study can make mathematics more alive, then here indeed is a promising path.

Our study of matrix algebra will involve the investigation of a significant posulational system, which will reflect the vigor of abstract mathematics. This is a unit in "hard" mathematics that has power and beauty. It will provide an effective language and some dynamic concepts that will enhance the student's ability to handle his first college courses yet not duplicate material.

Lastly, with the objective that the intellectually vigorous students may, in some small part, obtain an idea of what constitutes "mathematical research," there is appended a set of "Research Exercises." These are by no means overnight homework and any one of them may well constitute a project to be executed by several students. Such team operations are conducive to stimulating discourse and critical thinking.

CONTENTS

FOREWORD iii

PREFACE v

Chapter

1. MATRIX OPERATIONS 1
    - 1-1. Introduction 1
    - 1-2. The Order of a Matrix 4
    - 1-3. Equality of Matrices 8
    - 1-4. Addition of Matrices 10
    - 1-5. Addition of Matrices (Concluded) 19
    - 1-6. Multiplication of a Matrix by a Number 22
    - 1-7. Multiplication of Matrices 27
    - 1-8. Properties of Matrix Multiplication 40
    - 1-9. Properties of Matrix Multiplication (Concluded) 46
    - 1-10. Fields and Rings 55

2. THE ALGEBRA OF 2×2 MATRICES 61
    - 2-1. Introduction 61
    - 2-2. The Ring of 2×2 Matrices 65
    - 2-3. The Uniqueness of the Multiplicative Inverse 69
    - 2-4. The Inverse of a Matrix of Order Two 79
    - 2-5. The Determinant Function 85
    - 2-6. The Group of Invertible Matrices 93
    - 2-7. An Isomorphism between Complex Numbers and Matrices 103
    - 2-8. Algebras 112

3. MATRICES AND SYSTEMS OF LINEAR EQUATIONS 115
    - 3-1. Introduction 115
    - 3-2. The Triangulation Method 119
    - 3-3. Formulation in Terms of Matrices 125
    - 3-4. Solution by Means of Matrices 130
    - 3-5. The Diagonal Method 139
    - 3-6. Matrix Inversion 142
    - 3-7. Linear Systems in General 151

4. REPRESENTATION OF COLUMN MATRICES AS GEOMETRIC VECTORS 159
    - 4-1. The Algebra of Vectors 159
    - 4-2. Vectors and Directed Line Segments 162
    - 4-3. Geometrical Interpretations of the Multiplication of a Vector by a Number 167
    - 4-4. Geometrical Interpretation of the Addition of Two Vectors. 171
    - 4-5. The Inner Product of Two Vectors 177
    - 4-6. An Area and a Determinant 185
    - 4-7. The Interplay between Algebra and Geometry; Vector Analysis 188

Chapter

5. TRANSFORMATIONS OF THE PLANE 191
5-1. Vector Spaces and Subspaces 191
5-2. Functions and Geometric Transformations 199
5-3. Matrix Transformations 208
5-4. Linear Transformations 215
5-5. One-to-one Linear Transformations 221
5-6. Invariant Subspaces 227
5-7. Rotations and Reflections 233

APPENDIX: RESEARCH EXERCISES 241

BIBLIOGRAPHY 253

INDEX 255

Chapter 1

MATRIX OPERATIONS

1-1. Introduction

As we have studied more and more sophisticated mathematics, we have had occasion to use more and more sophisticated kinds of "numbers." We began with the positive whole numbers, 1, 2, 3,.... Then, in order to make subtractions like 3 − 7 possible, zero and the negative whole numbers, 0, −1, −2, −3,..., had to be introduced. Next, in order to make it possible to divide any number by any nonzero number, fractions like 1/2, −2/3, and −157/321 were invented. This did not bring us to the end of our story, for, in order that every positive number should have a square root, a cube root, a logarithm, etc., it was necessary to invent still more numbers: the infinite decimals or real numbers, such as 1.4142..., 3.1415928..., and 0.13131313.... Finally, in order that negative numbers should also have square roots, and that such quadratic equations as

$$x^2 + x + 1 = 0$$

should have solutions, it was necessary to invent complex numbers like $3 + 2i$, $1 + \pi i$, and $-1/2 + (1/37)i$.

Whenever there has seemed to be a good reason to do so, we have invented new sets of "numbers." For instance, in inventing complex quantities, we began not with the quantities themselves but with a purpose: to find a system of numbers each of which has a square root. When we have made one such invention, it is not hard to realize that there is no reason to stop inventing. Why should we not hope to invent many kinds of new numbers?

Of course, it is easy to invent things that do not work, but harder to invent things that do work — easy to invent things that are useless, but hard to invent things that are useful. The same is true of the invention of new kinds of numbers. The hard thing is to invent useful kinds of numbers, and kinds of numbers "that work." Nevertheless, a large number of more or less successful new kinds of numbers have been invented by mathematicians. In this book, we are going to study one of the most successful of these new kinds of numbers: the matrices.

Before we tell you what matrices are, it is well for us to emphasize their importance. They are useful in almost every branch of science and engineering. A great number of the computations made on the giant "electronic brains" are computations with matrices. Many problems in statistics are expressed in terms of matrices. Matrices come up in the mathematical problems of economics. They are extremely important in the study of atomic physics; indeed, atomic physicists express almost all their problems in terms of matrices, and it would not be an exaggeration to say that the algebra of matrices is the language of atomic physics. Many other kinds of algebra, like complex-number algebra and vector algebra, which some of you may already have studied, can be explained very easily in terms of matrices. So, in studying matrices, you will be studying one of the newest and most important, as well as one of the most interesting, branches of mathematics.

Let us look at a few simple examples.

Many a baseball fan, when he first opens the newspaper, refers to a tabulation similar to the following:

	G	AB	R	H
Aaron	68	280	52	109
Williams	52	194	29	60
Mantle	60	228	51	70
Lopez	63	241	38	72

If he is a Mantle fan, he looks at the entry in the third row and fourth column of numbers in order to learn how many hits Mantle has thus far obtained during the season.

You will note that we have said "row" in speaking of a horizontal array, and "column" in speaking of a vertical array. Thus, the third row is

$$60 \quad 228 \quad 51 \quad 70,$$

and the fourth column is

$$\begin{array}{r} 109 \\ 60 \\ 70 \\ 72 \end{array}$$

An assembler of TV sets might have before him a table of the following sort:

	Model A	Model B	Model C
Number of tubes	13	18	20
Number of speakers	2	3	4

This table indicates the number of tubes and the number of speakers used in assembling each model.

Omitting the row and column headings, let us focus our attention on the arrays of numbers in the last two examples:

$$\begin{array}{rrrr} 68 & 280 & 52 & 109 \\ 52 & 194 & 29 & 60 \\ 60 & 228 & 51 & 70 \\ 63 & 241 & 38 & 72 \end{array} \qquad \begin{array}{rrr} 13 & 18 & 20 \\ 2 & 3 & 4 \end{array}$$

Such arrays of *entries* are called *matrices* (singular: *matrix*). Thus a matrix is a rectangular array of entries appearing in rows and columns.

Actually, the entries may be complex numbers, functions, and in appropriate circumstances even matrices themselves; however, with a few exceptions that will be clearly indicated, we shall confine our attention to the real numbers with which we are already familiar.

Some examples of matrices are the following:

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & \sqrt{2} \\ 3.14 & 1 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 1/2 & 1/4 & 1/8 \end{bmatrix} . \qquad (1)$$

You will note here how square brackets [ ] are used in the mathematical designation of matrices.

A great advantage of this notation is the fact that we can use it in handling large sets of numbers as single units, thus simplifying the statement of complicated relationships.

1-2. The Order of a Matrix

The order of a matrix is given by stating first the number of rows and then the number of columns in the matrix. For example, the order of the matrices in the foregoing examples are respectively 2 × 3 (read "2 by 3"), 2 × 2, 4 × 1, and 1 × 3. Generally, a matrix that has m rows and n columns is called an $m \times n$ (read "m by n") matrix, or a matrix of order $m \times n$.

If the number of rows is the same as the number of columns, as in the second example above, then the matrix is square. Thus, given two linear equations in two unknowns,

$$\begin{aligned} 2x + 3y &= 7, \\ 1x - 2y &= 0, \end{aligned}$$

we observe that the coefficients of x and y constitute a square matrix:

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} .$$

When speaking of a square n × n matrix, we often refer to its order as n rather than n × n. For example, the 2 × 2 matrix

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

is a square matrix of order 2, and the 3 × 3 matrix

$$\begin{bmatrix} -1 & 2 & 3 \\ 4 & -5 & 6 \\ 7 & 8 & -9 \end{bmatrix}$$

is a square matrix of order 3.

If the number of rows is 1, as in the fourth example in (1), above, the matrix is called a row matrix or a row vector. For example, in terms of rectangular coordinates, a point in a plane might be designated by the row matrix $\begin{bmatrix} 2 & 3 \end{bmatrix}$, or a point in space by the row matrix $\begin{bmatrix} 2 & 3 & -1 \end{bmatrix}$.

Similarly, a column matrix or column vector is a matrix having just one column. Thus, the foregoing points can equally well be designated by column matrices,

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} ;$$

and the number of men, women, and children in a family might be denoted by

$$\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} .$$

Capital letters are often used to denote general matrices, and the corresponding small letters with appropriate subscripts are then employed to designate entries. Thus, we might have

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}.$$

In these examples, the entries located at the intersection of the 2nd row and 3rd column are denoted by a_{23} and b_{23}, respectively.

Generally, if the entry is located at the intersection of the i-th row and j-th column of matrix A, it is denoted by a_{ij}. An $m \times n$ matrix can be denoted compactly as $\left[a_{ij}\right]_{m \times n}$. Thus, the foregoing matrices A and B are

$$A = \left[a_{ij}\right]_{3 \times 3} \quad \text{and} \quad B = \left[b_{ij}\right]_{2 \times 3}.$$

If the order is clear from the context or is arbitrary, the notation might be reduced to

$$A = \left[a_{ij}\right] \quad \text{and} \quad B = \left[b_{ij}\right].$$

Associated with each matrix is another matrix called its transpose, which is often convenient to use and has interesting theoretical properties. The transpose A^t of a matrix A is formed by interchanging its rows and columns. For example, if

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -1 & 0 \end{bmatrix}, \quad \text{then} \quad A^t = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 2 & 0 \end{bmatrix}.$$

Definition 1-1. If $A = \left[a_{ij}\right]$ is an $m \times n$ matrix, then the transpose A^t of A is the $n \times m$ matrix $B = \left[b_{ij}\right]$ with $b_{ij} = a_{ji}$ for each i, j ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$).

Exercises 1-2

1. (a) Obtain from a newspaper or other similar source six examples of information presented in matrix form.

   (b) In each of your examples, state the order of the matrix.

   (c) In each of the examples, suggest an alternative method (not in matrix form) of presenting the same information.

2. A row vector with three entries can be used to tabulate a person's age, height, and weight.

   (a) Give a row vector that lists your age, height, and weight.

   (b) Suggest when it might be useful to employ such a vector.

3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 8 & 10 & 12 & 14 & 16 \\ -1 & -3 & -5 & 6 & 3 \\ 0 & 3 & -7 & 8 & 7 \end{bmatrix}.$$

   (a) What is the order of A?

   (b) Name the entries in the 4th row.

   (c) Name the entries in the 3rd column.

   (d) Name the entry a_{43}.

   (e) Name the entry a_{14}.

   (f) Name the entry a_{41}.

   (g) Write the transpose A^t.

4. Let

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) What is the order of B?

(b) Name the entries in the 3rd column.

(c) Name the entry b_{12}.

(d) For what values i, j is $b_{ij} \neq 0$?

(e) For what values i, j is $b_{ij} = 0$?

(f) Write the transpose B^t.

5. (a) Write a 3×3 matrix all of whose entries are whole numbers.

   (b) Write a 3×4 matrix none of whose entries are whole numbers.

   (c) Write a 5×5 matrix having all entries in its first two rows positive, and all entries in its last three rows negative.

6. (a) How many entries are there in a 2×2 matrix?

   (b) In a 3×3 matrix?

   (c) In an $n \times n$ matrix?

1-3. Equality of Matrices

Two matrices are equal provided they have the same order and each entry in the first is equal to the corresponding entry in the second. For example,

$$\begin{bmatrix} 1 & 4 & 0 \\ 2 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \times 2 & 2 - 2 \\ 4/2 & 16/2 & 8/2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2^1 \\ 2^2 \\ 2^3 \end{bmatrix}, \quad \begin{bmatrix} x^2 - 1 \\ x \end{bmatrix} = \begin{bmatrix} (x-1)(x+1) \\ x \end{bmatrix};$$

but

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \quad \begin{bmatrix} 0,0 \end{bmatrix} \neq \begin{bmatrix} 0 \end{bmatrix}.$$

Definition 1-2. Two matrices A and B are equal, $A = B$, if and only if they are of the same order and their corresponding entries are equal.

Thus,

$$\left[a_{ij}\right]_{m\times n} = \left[b_{ij}\right]_{m\times n}$$

if and only if $a_{ij} = b_{ij}$ for each i,j $(i = 1,2,\ldots,m;\quad j = 1, 2,\ldots,n)$.

Using the foregoing definition of equality, we can express certain relationships more compactly. For example, the equation

$$\begin{bmatrix} 2x + 3y \\ 3x - y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

can be employed instead of the two separate equations

$$\begin{aligned} 2x + 3y &= 7, \\ 3x - y &= 2; \end{aligned}$$

and

$$\begin{bmatrix} x + y & a + b \\ x - y & a - b \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$$

can be written in place of the four equations

$$\begin{aligned} x + y &= 5, & a + b &= -1, \\ x - y &= 1, & a - b &= 3. \end{aligned}$$

Exercises 1-3

1. Solve the following equations:

(a) $$\begin{bmatrix} x + 2 \\ 3 - y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

(b) $$\begin{bmatrix} x - 2y \\ x + y \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix},$$

(c) $$\begin{bmatrix} x^2 & y \\ x & y^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

2. From the equalities of the matrices $A = B$ and $B = C$, would you conclude that $A = C$? Why?

3. Write the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

if

$$a_{ij} = 2i + 3j - 4.$$

4. Write the matrix whose entries are the sums of the corresponding entries of the matrices

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -3 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 2 \\ -3 & 4 \\ 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

5. Write the matrix whose entries are the differences (first minus second) of the corresponding entries in Exercise 4.

1-4. Addition of Matrices

We have now defined matrices and studied some of their most elementary properties. But we have not really made them work. To do this, we must give rules for adding and multiplying matrices, just as was done with complex numbers. If these numbers were defined bluntly as expressions of the form $a + bi$, without the operations of addition and multiplication, and without relation to the solution of such equations as

$$x^2 + x + 1 = 0,$$

they would be of relatively little interest. What gives life to complex numbers is the fact that we are able to define addition and multiplication for them in such a way that we have a whole algebra of complex numbers, which is indeed useful and interesting.

The same remark applies to matrices. To give the study of matrices its real content, we must define "sum" and "product" for matrices. In this section, we define and study sums of matrices. Products will be considered later.

You will recall that when two complex numbers are added, for example $3 + 5i$ and $-2 + 4i$, the two real components and the two imaginary components are added separately. Thus,

$$(3 + 5i) + (-2 + 4i) = (3 + (-2)) + (5 + 4)i = 1 + 9i.$$

If we represent the complex numbers as column vectors, we find their sum by adding corresponding entries; thus,

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}.$$

This suggests the pattern used in adding matrices of the same order. The sum of two such matrices is obtained by adding the individual entries in corresponding positions. For example,

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 2 & 1 \\ 1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 2 \\ 0 & 3 & 2 \end{bmatrix}.$$

Since we shall not even give a rule by which matrices of different orders could be added, we shall add two matrices only if they are of the same order. Two matrices that have the same order are said to be conformable for addition. The sum has the same order as the two addends.

Definition 1-3. The sum $A + B$ of two $m \times n$ matrices A and B is the $m \times n$ matrix C such that the element c_{ij} in the i-th row and j-th column of C is equal to the sum $a_{ij} + b_{ij}$ of the elements a_{ij} and b_{ij} in the i-th row and j-th column of A and B, respectively.

Thus,

$$\left[a_{ij}\right]_{m \times n} + \left[b_{ij}\right]_{m \times n} = \left[a_{ij} + b_{ij}\right]_{m \times n} .$$

For instance,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} .$$

If we consider all $m \times n$ matrices, with m and n fixed, as constituting a set $S_{m,n}$, and if A and B are elements of $S_{m,n}$, then $A + B$ is also an element of this set. That is, if $A \in S_{m,n}$ (read "A is an element of $S_{m,n}$") and $B \in S_{m,n}$, then $(A + B) \in S_{m,n}$.

In the algebra of real numbers R, the equation

$$a + 0 = a$$

is satisfied for all $a \in R$ (this time, read "for all $a \in R$" as "for all elements a of R"). Accordingly, we say that 0 is the identity element for addition in R. In the algebra of matrices, the matrices all of whose entries are 0 play a corresponding role. Thus,

$$\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 + 0 & 3 + 0 \\ -1 + 0 & 4 + 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} .$$

Such a matrix is called a zero matrix and is denoted by $\underline{0}$. If the order $m \times n$ is significant, we write $0_{m \times n}$; or, if the matrix is square, we might write 0_n, where n indicates the order of the matrix. Thus,

$$0_{1\times 2} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad 0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad 0_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The equation

$$A_{m\times n} + 0_{m\times n} = A_{m\times n}$$

clearly is valid for all $A_{m\times n}$.

The addition of matrices is a commutative operation, as we can readily verify. Thus,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

In particular, the sum of the two matrices on the left is a matrix having $a_{12} + b_{12}$ as element in the first row and second column, and the corresponding element of the sum on the right is $b_{12} + a_{12}$. But

$$a_{12} + b_{12} = b_{12} + a_{12},$$

by the commutative law for the addition of real numbers.

The foregoing observation holds generally, of course, so that we have the following result:

Theorem 1-1. If the matrices A and B are conformable for addition, then they satisfy the commutative law for addition:

$$A + B = B + A.$$

Proof. We have

$$
\begin{aligned}
A + B &= \begin{bmatrix} a_{ij} \end{bmatrix} + \begin{bmatrix} b_{ij} \end{bmatrix} \\
&= \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix} \\
&= \begin{bmatrix} b_{ij} + a_{ij} \end{bmatrix} \\
&= \begin{bmatrix} b_{ij} \end{bmatrix} + \begin{bmatrix} a_{ij} \end{bmatrix} \\
&= \quad B + A.
\end{aligned}
$$

Thus, in terms of our usual notation, the element in the i-th row and j-th column of the sum on the left is $a_{ij} + b_{ij}$, and the corresponding element of the sum on the right is $b_{ij} + a_{ij}$. But

$$a_{ij} + b_{ij} = b_{ij} + a_{ij},$$

by the commutative law for the addition of real numbers; hence the theorem follows from the definition (Definition 1-2) of the equality of two matrices.

The addition of conformable matrices is also associative; that is,

$$A + (B + C) = (A + B) + C.$$

For example,

$$
\begin{aligned}
&\begin{bmatrix} 2 & 3 & 1 \\ -4 & 0 & 6 \end{bmatrix} + \left(\begin{bmatrix} -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 4 \\ 5 & 1 & 2 \end{bmatrix} \right) \\
&= \begin{bmatrix} 2 & 3 & 1 \\ -4 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 5 \\ -1 & 1 & 9 \end{bmatrix},
\end{aligned}
$$

and also

$$
\begin{aligned}
&\left(\begin{bmatrix} 2 & 3 & 1 \\ -4 & 0 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \right) + \begin{bmatrix} 1 & 0 & 4 \\ 5 & 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 5 & 1 \\ -6 & 0 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 4 \\ 5 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 5 \\ -1 & 1 & 9 \end{bmatrix}.
\end{aligned}
$$

We can state the associative property as a theorem and prove it, as follows:

Theorem 1-2. If the matrices A, B, and C are conformable for addition, then they satisfy the associative law for addition:

$$A + (B + C) = (A + B) + C.$$

Proof. We note that, in terms of our usual notation, the element in the i-th row and j-th column of the sum on the left is $a_{ij} + (b_{ij} + c_{ij})$, and the corresponding element of the sum on the right is $(a_{ij} + b_{ij}) + c_{ij}$. But

$$a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}.$$

You can complete the proof of Theorem 1-2 by telling why this last equality is valid for all real numbers a_{ij}, b_{ij}, and c_{ij}, and why this equality implies the matrix equality

$$A + (B + C) = (A + B) + C.$$

Since it is immaterial in which order the matrices are added, we write $A + B + C$ for either expression:

$$A + (B + C) = (A + B) + C = A + B + C.$$

Once we know how to add numbers, it is usual to consider subtraction. You will recall that the negative, or additive inverse, of the real number a is denoted by $-a$. It satisfies the equation

$$a + (-a) = 0.$$

Subtraction of matrices arises in a similar manner.

Definition 1-4. Let A be an $m \times n$ matrix. Then the negative of A, written $-A$, is the $m \times n$ matrix each of whose entries is the negative of the corresponding entry of A.

Definition 1-5. If A and B are two $m \times n$ matrices, then the difference of A and B, designated by $A - B$, is the sum of the matrices A and the negative of B.

Thus, for $A + (-B)$, where A and B are matrices of equal orders, we write $A - B$ and say that the symbols indicate that B is to be subtracted from A. For example,

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -2 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 5 \\ 1 & -4 & -2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & t-c \\ t+c & 4 \end{bmatrix} - \begin{bmatrix} 1 & t \\ c & 2 \end{bmatrix} = \begin{bmatrix} 0 & -c \\ t & 2 \end{bmatrix}.$$

Now we can easily prove the following theorem:

Theorem 1-3. If A and B are $m \times n$ matrices, then

(a) $A + (-A) = \underline{0}$,

(b) $-(-A) = A$,

(c) $-\underline{0} = \underline{0}$,

(d) $-(A + B) = (-A) + (-B)$.

Proof of Theorem 1-3 (a). The entry in the i-th row and j-th column of

$-A$ is, by definition, $-a_{ij}$. Thus the entry in the i-th row and j-th column of $A + (-A)$ is $a_{ij} + (-a_{ij})$. But $a_{ij} + (-a_{ij}) = 0$. Hence, every entry of $A + (-A)$ is zero; that is, $A + (-A)$ is the zero matrix.

The proofs of the remaining parts are similar and are left to the student as exercises.

Exercises 1-4

1. Find values x, y, a, and b that satisfy the matrix relationship

$$\begin{bmatrix} x+3 & 2y-8 \\ a+1 & 4x+6 \\ b-3 & 3b \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ -3 & 2x \\ 2b+4 & -21 \end{bmatrix} .$$

2. If

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -5 & 6 \\ 0 & 8 & -3 \\ 4 & 6 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 & 8 \\ -2 & 6 & -1 \\ 0 & 2 & 3 \\ 4 & -1 & 8 \end{bmatrix} ,$$

determine the entry in the sum $A + B$ that is at the intersection of

(a) the 3rd row and 2nd column,

(b) the 1st row and 3rd column,

(c) the 4th row and 1st column.

3. Compute

$$\begin{bmatrix} 1/2 & 1/3 \\ 1/4 & 1/5 \end{bmatrix} - \begin{bmatrix} 1/6 & 1/7 \\ 1/8 & 1/9 \end{bmatrix} .$$

4. Compute

$$\begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 1/5 & 1/6 & 1/7 \\ 1/8 & 1/9 & 1/10 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Compute

$$\begin{bmatrix} x & y & z \\ p & s & t \\ u & v & w \end{bmatrix} + \begin{bmatrix} 1-x & -y & -z \\ -p & 1-s & -t \\ -u & -v & 1-w \end{bmatrix}.$$

6. (a) Does the sum

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} + 0_2$$

make sense?

(b) Does the sum

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} + 0_3$$

make sense?

(c) What is the latter sum?

7. Compute

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \sqrt{2} & 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 17 & 8 \\ 9 & 6 & 14 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 3 \\ 14 & 8 & 6 \\ 1+\sqrt{2} & 11 & 11 \end{bmatrix}.$$

8. Compute

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{bmatrix} .$$

9. Given

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} , \quad B = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 0 & 1 \end{bmatrix} , \quad \text{and} \quad C = \begin{bmatrix} 4 & 2 \\ 1 & 0 \\ -2 & -4 \end{bmatrix} ,$$

compute the following:

(a) $A + B$,

(b) $A + (B + C)$,

(c) $A - B$,

(d) $(A - B) + C$,

(e) $(A + B) + C$,

(f) $B - A$.

10. (a) In Exercise 9, consider the answers to parts (b) and (e). What law is illustrated?

(b) In Exercise 9, consider the answers to parts (c) and (f). What conclusion can be drawn?

11. Prove Theorem 1-3 (b).

12. Prove Theorem 1-3 (c).

13. Prove Theorem 1-3 (d).

14. Assuming that A and B are conformable for addition, prove that $A^t + B^t = (A + B)$.

1-5. Addition of Matrices (Concluded)

The theorems given in Section 1-4 include exact analogues of all the basic laws of ordinary algebra, insofar as these laws refer to addition and subtraction.

We know that all of the more complicated algebraic laws concerning addition and subtraction are consequences of these basic laws. Since the basic laws of the addition and subtraction of matrices are the same as the basic laws of the addition and subtraction of ordinary algebra, all the other laws for the addition and subtraction of matrices must be the same as the corresponding laws for the addition and subtraction of numbers. We can state this as follows:

Insofar as only addition and subtraction are involved, the algebra of matrices is exactly like the ordinary algebra of numbers.

So you do not have to study the algebra of addition and subtraction of matrices – you already know it! But now the algebra that you already know has a new and much richer content. Formerly, it could be applied only to numbers. Now, it can be applied to matrices of any order. Thus, we have obtained a very considerable result with a very small effort, simply by observing that our old algebraic laws of addition and subtraction apply not only to numbers, but also to quite different kinds of things, namely, matrices. This very powerful trick of putting old results in new settings has been used many times, and often with great success, in the most modern mathematics.

A good example of the general principle emphasized above is provided by the following problem. Suppose that A and B are known matrices of the same order. How can we solve the equation

$$X + A = B$$

for the unknown matrix X? The answer is easy. We do exactly what we learned to do with numbers. Add the matrix $-A$ to both sides. This gives

$$X + A + (-A) = B - A.$$

Since $A + (-A) = \underline{0}$, and $X + \underline{0} = X$, we have

$$X = B - A.$$

This is our solution.

Exercises 1-5

1. Solve the equation

$$X + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for the matrix X.

2. Solve the equation

$$X + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 3 \\ 4 & 3 & 4 \end{bmatrix}$$

for the matrix X.

3. If $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} - \begin{bmatrix} -6 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 2 & -3 \end{bmatrix}$, determine $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$.

4. If

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \text{ determine } \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

5. If

$$\begin{bmatrix} 2 & -3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 5 & -1 \end{bmatrix},$$

determine x_1, x_2, y_1, and y_2.

6. Prove that if the matrices A, B, and C are conformable for addition, then $(A + C) - (A + B) = C - B$.

7. Is the equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

valid?

1-6. Multiplication of a Matrix by a Number

Once we know how to add numbers, it is customary to define $2x$ as the sum $x + x$, $3x$ as the sum $2x + x$, etc. Fractional parts of x are defined by requiring that $(1/2)x + (1/2)x = x$, $(1/3)x + (1/3)x + (1/3)x = x$, etc. All of this can readily be done with matrices. If we add two equal matrices, the sum is clearly a matrix in which each entry is exactly twice the corresponding entry in the two given matrices. Thus

$$\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 2(2) & 2(3) \\ 2(-1) & 2(0) \end{bmatrix}.$$

Likewise, for three equal matrices we have

$$\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 3(2) & 3(3) \\ 3(-1) & 3(0) \end{bmatrix}.$$

Each of the above sums may be considered to be the product of a number and a matrix. We write

$$2\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -2 & 0 \end{bmatrix},$$

$$3\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ -3 & 0 \end{bmatrix}.$$

The equation

$$\frac{1}{2} A + \frac{1}{2} A = A,$$

defining the matrix $(1/2)A$, is clearly satisfied by the matrix each of whose entries is exactly $1/2$ the corresponding entry of A; the equation

$$\frac{1}{3} A + \frac{1}{3} A + \frac{1}{3} A = A,$$

defining the matrix $(1/3)A$, is clearly satisfied by the matrix each of whose entries is exactly $1/3$ the corresponding entry of A.

These considerations lead us to make the following general definition.

Definition 1–6. The product $cA = Ac$ of a number c and an $m \times n$ matrix A is the $m \times n$ matrix B such that the element b_{ij} in the i-th row and j-th column of B is equal to the product ca_{ij} of the number c and the entry a_{ij} in the i-th row and j-th column of A.

Thus,

$$c\begin{bmatrix}a_{ij}\end{bmatrix}_{m \times n} = \begin{bmatrix}a_{ij}\end{bmatrix}_{m \times n} c = \begin{bmatrix}ca_{ij}\end{bmatrix}_{m \times n}.$$

For example,

$$c \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \\ ca_{31} & ca_{32} \end{bmatrix} .$$

Note that here we have defined the product of a matrix by a number, not the product of two matrices. It is possible also to define the product of two matrices; this will be done in Section 1–7.

Now we may state the following theorem about products of matrices by numbers.

Theorem 1-4. If A and B are $m \times n$ matrices, and x and y are numbers, then

(a) $x(yA) = (xy)A,$

(b) $(x+y)A = XA + yA,$

(c) $(-1)A = -A,$

(d) $x(A + B) = xA + xB,$

(e) $x\,\underline{0} = \underline{0},$

(f) $0\,A = \underline{0}.$

Part (e) states that the product of a number and the zero matrix is the zero matrix, and part (f) states that the product of the zero number and any matrix is the zero matrix.

Proof of Theorem 1-4 (d). The entry in the i-th row and j-th column of the matrix $A + B$ is $a_{ij} + b_{ij}$. The entry in the i-th row and j-th column of matrix $x(A + B)$ is therefore, by definition, $x(a_{ij} + b_{ij})$. Now the entry in the i-th row and j-th column of the matrix xA is $x\,a_{ij}$; that in the i-th row and j-th column of the matrix xB is $x\,b_{ij}$. Thus the entry in the i-th row and j-th column of the matrix $xA + xB$ is $x\,a_{ij} + x\,b_{ij}$. Since the entries are numbers and, for all numbers, $a(b + c) = ab + ac$, we have

$$x(a_{ij} + b_{ij}) = x\,a_{ij} + x\,b_{ij},$$

so that each entry in the matrix $x(A + B)$ is the same as the corresponding entry of the matrix $xA + xB$. Hence,

$$x(A + B) = xA + xB.$$

The other parts of the above theorem may be proved in a similar way.

When we studied the laws governing the addition and subtraction of matrices, we saw that they were parallel to the laws governing addition and subtraction in ordinary algebra. The situation when we come to the multiplication of matrices by numbers is rather similar, but not exactly the same. The various parts of Theorem 1-4 resemble the basic algebraic laws for multiplication very closely. Thus, many of the more complicated ordinary algebraic laws and procedures governing multiplication still remain correct for expressions involving the multiplication of matrices by numbers. The difference is that the product of a number by a number is a number, but the product of a matrix by a number is not a number but a matrix.

We are now able to solve some matrix equations involving addition, subtraction, and multiplication by a number. Let us look at an example.

Suppose we want to solve the equation

$$-2\left(X + \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\right) = 3X + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We first perform the indicated multiplication by −2, in accordance with part (d) of the above theorem, to get

$$-2X + \begin{bmatrix} -2 & -4 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & -2 \end{bmatrix} = 3X + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then we add $2X$ to both sides of the equation to obtain

$$\begin{bmatrix} -2 & -4 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & -2 \end{bmatrix} = 3X + 2X + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Next we use part (b) of the theorem to find that $3X + 2X = 5X$, so that

$$\begin{bmatrix} -2 & -4 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & -2 \end{bmatrix} = 5X + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Adding

$$- \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to both sides, we find that

$$\begin{bmatrix} -3 & -4 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & -3 \end{bmatrix} = 5X.$$

Multiplying both sides of this last equation by 1/5, we see by part (a) of the theorem that

$$X = \begin{bmatrix} -3/5 & -4/5 & -6/5 \\ 0 & -2/5 & -4/5 \\ 0 & 0 & -3/5 \end{bmatrix} .$$

This is our solution.

Exercises 1-6

1. For

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 5 \\ 6 & 9 & -1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 5 & -1 & 0 \\ 7 & 8 & -1 \end{bmatrix},$$

determine the result of the following operations:

(a) $2A - B + C$,
(b) $3A - 4B - 2C$,
(c) $7A - 2(B - C)$,
(d) $3(A - 2B + 3C)$.

2. For

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{bmatrix},$$

determine the result of the following operations:

(a) $2A - B + C$,
(b) $3A - 6B + 9C$,
(c) $7A - 2(B - C)$,
(d) $3(A - 2B + 3C)$.

3. Let A, B, and C be the matrices of Exercise 2. Solve the equation

$$\frac{1}{2}(X + A) + 3(X + (2X + B)) + C,$$

giving all the steps in detail, and justifying each step.

4. Let A, B, and C be the matrices of Exercise 2. Solve the equation

$$2(X + B) = 3(X + (X/2 + A)) + C.$$

5. Prove Theorem 1-4 (a).

6. Prove Theorem 1-4 (b).

1-7. Multiplication of Matrices

Thus far, we have defined and studied the addition and subtraction of matrices and the multiplication of a matrix by a number. We still have not defined the product of two matrices. Since the formal definition is somewhat complicated and may at first seem odd, let us look at a simple practical problem that will lead us to operate with two matrices in the way that we shall ultimately call multiplication.

In Section 1—1, the number of tubes and the number of speakers used in assembling three different models of TV sets were specified by a table:

	Model A	Model B	Model C
Number of tubes	13	18	20
Number of speakers	2	3	4

This array will be called the parts—per—set matrix.

Suppose orders were received in January for 12 sets of model A, 24 sets of model B, and 12 sets of model C; and in February for 6 sets of model A, 12 of model B, and 9 of model C. We can arrange the information in the form of a matrix:

	January	February
Model A	12	6
Model B	24	12
Model C	12	9

This will be called the sets—per—month matrix.

To determine the number of tubes and speakers required in each of the months for these orders, it is clear that we must use both sets of information. For instance, to compute the number of tubes needed in January, we multiply each entry in the 1st row of the parts—per—set matrix by the corresponding entry in the 1st column of the sets—per—month matrix, and then add the three products. Thus, the number of tubes required in January is

$$13(12) + 18(24) + 20(12) = 828.$$

To compute the number of speakers needed in January, we multiply each entry in the 2nd row of the parts—per—set matrix by the corresponding entry in the 1st column of the sets—per—month matrix and then add the products. Thus, the number of speakers for January is

$$2(12) + 3(24) + 4(12) = 144.$$

For February, first we multiply the entries from the 1st row of the parts—per—set matrix by the corresponding entries from the 2nd column of the sets—per—month matrix and add to determine the number of tubes; secondly, we multiply the entries from the 2nd row of the parts—per—set matrix by the corresponding entries from the 2nd column of the sets—per—month matrix and add to determine the number of speakers. Thus the numbers of tubes and speakers for February are, respectively,

$$13(6) + 18(12) + 20(9) = 474,$$

and

$$2(6) + 3(12) + 4(9) = 84.$$

We can arrange the four sums in an array, which we shall call the parts—per—month matrix:

	January	February
Number of tubes	828	474
Number of speakers	144	84

Can we now represent our "operation" in equation form? Let us try:

$$\begin{bmatrix} 13 & 18 & 20 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 12 & 6 \\ 24 & 12 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 828 & 474 \\ 144 & 84 \end{bmatrix} . \qquad (1)$$

We have "multiplied" the parts-per-set matrix by the sets-per-month matrix to get just what should be expected, the parts-per-month matrix!

Note that, in Equation (1), 828 equals the sum of the products of the entries in the 1st row of the left-hand factor by the corresponding entries in the 1st column of the right-hand factor. Likewise, 474 equals the sum of the products of the entries in the 1st row of the left-hand factor by the corresponding entries in the 2nd column of the right-hand factor, and so on. Consider the "product" matrix

$$\begin{bmatrix} 828 & 474 \\ 144 & 84 \end{bmatrix}$$

in the symbolic form,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} .$$

The subscripts indicate the row and column in which the entry appears; they also indicate the row and the column of the two factor matrices that are combined to get that entry. Thus, the entry a_{21} in the 2nd row and 1st column is found by adding the products formed when the entries in the 2nd row of the left-hand factor are multiplied by the corresponding entries in the 1st column of the right-hand factor. The most concise description of the process is: "Multiply row by column."

The description, "Multiply row by column," of the pattern in the foregoing simple practical problem serves as our guide in establishing the general rule for the multiplication of two matrices. Very simply the rule is to multiply entries of a row by corresponding entries of a column and then add the products. Thus, given two matrices A and B, to find the entry in the i-th row and j-th column of the product matrix AB, multiply each entry in the i-th row of the left-hand factor A by the corresponding entry in the j-th column of the right-hand factor B, and then add all the resulting terms. Since there must be an entry in each row of the left-hand factor to match with each entry in a column of the right-hand factor, and conversely, the product is not defined unless the number of columns in the left-hand factor is equal to the number of rows in the right-hand factor. When the number of columns in the left-hand factor equals the number of rows in the right-hand factor, the matrices are conformable for multiplication.

A diagram can aid understanding; see Figure 1-1.

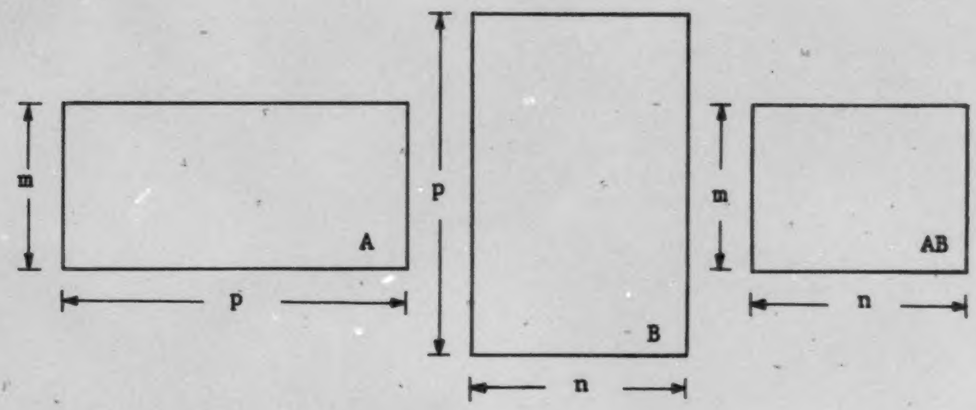


Figure 1-1. Matrices A and B that are conformable for multiplication. The number of columns of A must be equal to the number of rows of B. Then the product AB has the same number of rows as A and the same number of columns as B.

An entry in the product AB is found by multiplying each of the n entries in a row of A by the corresponding one of the n entries in the column of B and taking the sum; see Figure 1-2.

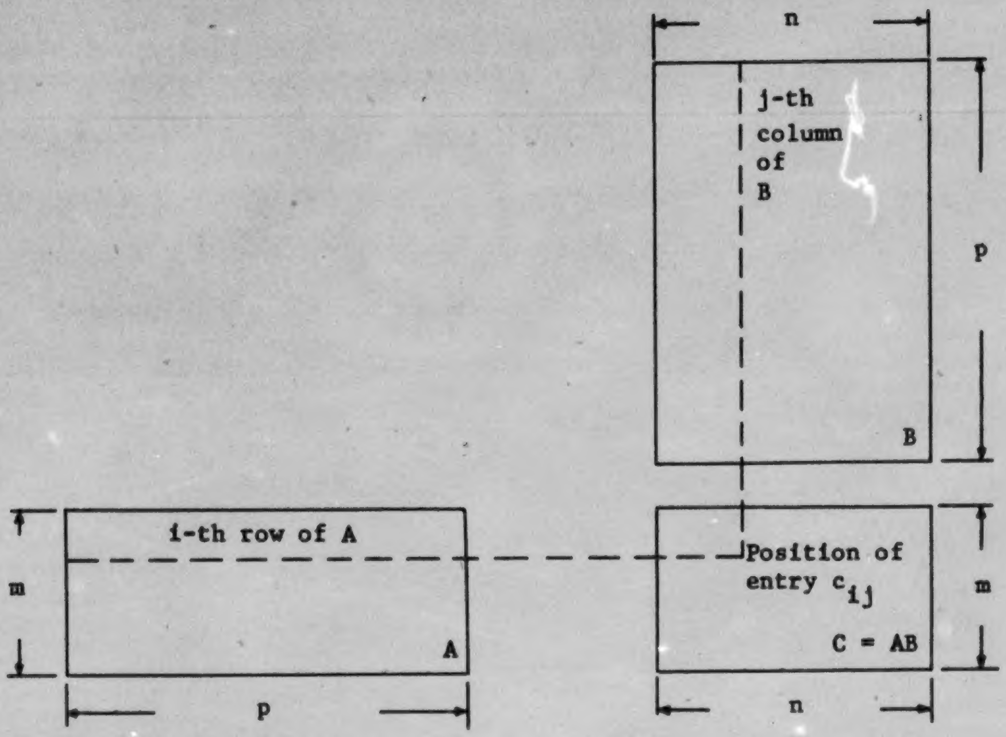

n
j-th
column
of
B
p
B
i-th row of A
m
A
p
Position of
entry c_{ij}
C = AB
m
n

Figure 1–2. Determination of an entry in the product AB of matrices A and B that are conformable for multiplication.

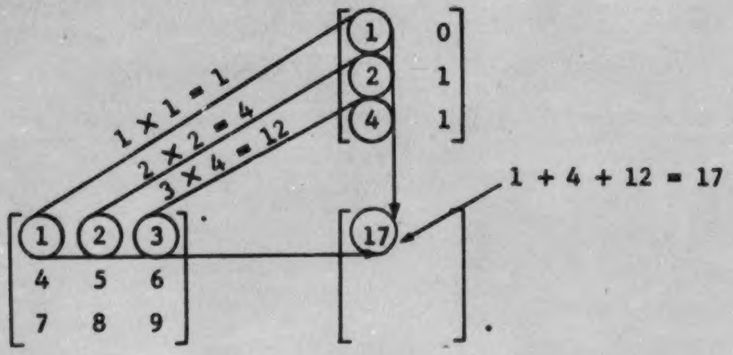
Thus, to multiply

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \text{by} \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 4 & 1 \end{bmatrix},$$

we first write

$$\begin{array}{cc} & \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 4 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} & \begin{bmatrix} & \\ & \\ & \end{bmatrix}. \end{array}$$

To determine the entry in the 1st row and 1st column of the product AB, we compute as follows:



Determining one entry of the product after another in this way, we finally obtain the complete answer for the product AB:

$$\begin{array}{cc} & \begin{bmatrix}1 & 0\\ 2 & 1\\ 4 & 1\end{bmatrix} \\ \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} & \begin{bmatrix}17 & 5\\ 38 & 11\\ 59 & 17\end{bmatrix}\end{array} .$$

(Check each of the entries of the answer yourself!) That is,

$$AB = \begin{bmatrix}17 & 5\\ 38 & 11\\ 59 & 17\end{bmatrix} .$$

To get the answer, 18 multiplications and 12 additions of pairs of numbers are necessary.

Although it may be a bit confusing at first, we place the factors adjacent to each other in the following examples since this is the arrangement usually employed:

(a) $$\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(1) + 2(4) & 1(2) + 2(0) & 1(3) + 2(1) \\ 3(1) + 1(4) & 3(2) + 1(0) & 3(3) + 1(1) \\ -1(1) + 2(4) & -1(2) + 2(0) & -1(3) + 2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 & 5 \\ 7 & 6 & 10 \\ 7 & -2 & -1 \end{bmatrix},$$

(b) $$\begin{bmatrix} 1 & 7 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(2) + 7(4) + 3(1) \end{bmatrix} = \begin{bmatrix} 33 \end{bmatrix},$$

(c) $$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(7) & 2(3) \\ 4(1) & 4(7) & 4(3) \\ 1(1) & 1(7) & 1(3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 14 & 6 \\ 4 & 28 & 12 \\ 1 & 7 & 3 \end{bmatrix}.$$

Let us now proceed to define multiplication formally.

Definition 1-7. Let

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times p} \quad \text{and} \quad B = \begin{bmatrix} b_{jk} \end{bmatrix}_{p \times n}$$

be matrices of order $m \times p$ and $p \times n$, respectively. The product AB is the matrix of order $m \times n$, of which the element in the i-th row and the j-th column is the sum of the products formed by multiplying elements of the i-th row of A by corresponding elements of the j-th column of B.

The definition of the product of two matrices can be expressed in terms of the "$\sum$ notation" for sums. Recall that, in the "$\sum$ notation," we write the sum

$$S = X_1 + X_2 + \cdots + X_n$$

of n numbers as

$$S = \sum_{k=1}^{n} X_k .$$

In this notation, the sum

$$a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

is expressed as

$$\sum_{k=1}^{n} a_{ik} b_{kj} .$$

This notation enables us to express Definition 1–7 more compactly:

Definition 1–7'. Let

$$A = \left[a_{ij}\right]_{m\times p} \quad \text{and} \quad B = \left[b_{jk}\right]_{p\times n}$$

be matrices of order $m \times p$ and $p \times n$, respectively. The product AB is the matrix of order $m \times n$, given by

$$AB = \left[a_{ij}\right]_{m\times p} \left[b_{jk}\right]_{p\times n} = \left[\left(\sum_{j=1}^{p} a_{ij} b_{jk} \right) \right]_{m\times n} = \left[c_{ik}\right]_{m\times n} .$$

Note that we have defined the product of two matrices only when the number of columns of the left-hand factor is the same as the number of rows of the right-hand factor. Also note that the number of rows in the product is the same as the number of rows in the left-hand factor, and that the number of columns in the product is the same as the number of columns in the right-hand factor.

Exercises 1-7

1. Let

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 11 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 0 & 1 \end{bmatrix}, \text{ and } D = \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -3 & -3 \end{bmatrix}.$$

State the orders of each of the following matrices:

(a) AB, (e) BD,

(b) DA, (f) D(AB),

(c) AD, (g) (CB)(DA),

(d) CB, (h) B(DA).

2. Perform the following matrix multiplication, where possible:

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$,

(c) $\begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 6 & 1 \\ 3 & 5 \end{bmatrix}$,

(d) $\begin{bmatrix} 4 & 2 \\ 6 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 0 \end{bmatrix}$,

(e) $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 2 & -1 & 6 \end{bmatrix}$,

(f) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & -1 & 6 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$.

3. Let $X = \begin{bmatrix} 2 & -2 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$,

$$U = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} , \text{ and } W = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} .$$

Compute the following:

(a) $5UX$,
(b) $(5W)(3Y)$,
(c) $5XU - (2X - Y)W$,
(d) $(U - W)(X + Y)$,
(e) $XU + YW$,
(f) $(X - Y)(U - W)$.

4. Perform the following matrix multiplications:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 2 & 0 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$,

(c) $$\begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

(d) $$\begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_1 \end{bmatrix},$$

(e) $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix},$$

(f) $$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(g) $$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}.$$

5. If

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

test the rule that $(AB)C = A(BC)$.

6. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} .$$

Compute

(a) AB,

(b) AB^t,

(c) BB^t,

(d) $(AB)B^t$,

(e) $A(BB^t)$,

(f) $A(B + B^t)$,

(g) $A(B - B^t)$,

(h) $AB - AB^t$,

(i) $AA - BB + B^tB^t$,

(j) $(AA)A$.

7. Let I denote the identity matrix of order 3 (see page 52):

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Let A and B be as in Exercise 7. Compute AI, BI, and B^tI. Compute $(AI)I$ and $((AI)I)I$.

8. Let

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} .$$

Find $(AB)^t$ and $B^t A^t$.

9. For a certain manufacturing plant, the following information is given:

	Part 1	Part 2	Part 3
Cost	2	3	5

	Subassembly 1	Subassembly 2
Part 1	4	1
Part 2	3	5
Part 3	7	2

	Model 1	Model 2	Model 3
Subassembly 1	2	1	2
Subassembly 2	3	4	5

	Day 1	Day 2	Day 3
Model 1	7	8	8
Model 2	3	4	5
Model 3	3	5	6

Determine the parts-per-model matrix and the cost-per-day matrix.

1-8. Properties of Matrix Multiplication

We have learned that insofar as only addition and subtraction are involved, the algebra of matrices is exactly like the ordinary algebra of numbers; see Section 1-5. At this moment, we might be concerned about multiplication since the definition seems a bit unusual. Is the algebra of matrices like the ordinary algebra of numbers insofar as multiplication is concerned?

Let us consider an example that will yield an answer to the foregoing question. Let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

If we compute AB, we find $AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Now, if we reverse the order of the factors and compute BA, we find

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus AB and BA are different matrices!

For another example, let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(1) + 2(4) & 1(2) + 2(0) & 1(3) + 2(1) \\ 3(1) + 1(4) & 3(2) + 1(0) & 3(3) + 1(1) \\ -1(1) + 2(4) & -1(2) + 2(0) & -1(3) + 2(1) \end{bmatrix} = \begin{bmatrix} 9 & 2 & 5 \\ 7 & 6 & 10 \\ 7 & -2 & -1 \end{bmatrix}$$

while

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1(1) + 2(3) + 3(-1) & 1(2) + 2(1) + 3(2) \\ 4(1) + 0(3) + 1(-1) & 4(2) + 0(1) + 1(2) \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 3 & 10 \end{bmatrix}.$$

Again AB and BA are different matrices; they are not even of the same order!

Thus we have a first difference between matrix algebra and ordinary algebra, and a very significant difference it is indeed. When we multiply numbers, we can rearrange factors since the commutative law holds: For all $x \in R$ and $y \in R$, we have $xy = yx$. When we multiply matrices, we have no such law and we must consequently be careful to take the factors in the order given. We must consequently distinguish between the result of multiplying B on the right by A to get BA, and the result of multiplying B on the left by A to get AB. In the algebra of numbers, these two operations of "right multiplication" and "left multiplication" are the same; in matrix algebra, they are not necessarily the same.

Let us explore some more differences! Let

$$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}.$$

Patently, $A \neq \underline{0}$ and $B \neq \underline{0}$. But if we compute AB, we obtain

$$\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$$

thus, we find $AB = \underline{0}$. Again, let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The second major difference between ordinary algebra and matrix algebra is that the product of two matrices can be a zero matrix without either factor being a zero matrix.

The breakdown for matrix algebra of the law that $xy = yx$ and of the law that $xy = 0$ only if either x or y is zero causes additional differences.

For instance, for real numbers we know that if $ab = ac$, and $a \neq 0$, then $b = c$. This property is called the cancellation law for multiplication.

Proof. We divided the proof into simple steps:

(a) $ab = ac$,

(b) $ab - ac = 0$,

(c) $a(b - c) = 0$,

(d) $b - c = 0$,

(e) $b = c$.

For matrices, the above step (d) fails and the proof is not valid. In fact, AB can be equal to AC, with $A \neq \underline{0}$, yet $B \neq C$. Thus, let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & -7 \end{bmatrix} = AC,$$

and

$$A \neq \underline{0},$$

but

$$B \neq C.$$

Let us consider another difference. We know that a real number a can have at most two square roots; that is, there are at most two roots of the equation $xx = a$.

Proof. Again, we give the simple steps of the proof:

(a) Suppose that $yy = a$; then

(b) $xx = yy$,

(c) $xx - yy = 0$,

(d) $(x - y)(x + y) = xx + (-y\ x + x\ y) - yy$,

(e) $yx = xy$.

(f) From steps (c) and (d), $(x - y)(x + y) = xx - yy$.

(g) From steps (e) and (b), $(x - y)(x + y) = 0$.

(h) Therefore, either $x - y = 0$ or $x + y = 0$.

(i) Therefore, either $x = y$ or $x = -y$.

For matrices, step (e) and step (h) fail. Therefore, the foregoing proof is not valid if we try to apply it to matrices. In fact, it is false that a matrix can have at most two square roots: We have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus the matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has the four different square roots

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad K = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

There are more! Given any number $x \neq 0$, we have

$$\begin{bmatrix} 0 & x \\ 1/x & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 1/x & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By giving x any one of an infinity of different real values, we obtain an infinity of different square roots of the matrix I:

$$\begin{bmatrix} 0 & 2 \\ 1/2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1/3 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -4 \\ -1/4 & 0 \end{bmatrix}, \text{ etc.}$$

Thus the very simple 2 × 2 matrix I has infinitely many distinct square roots! You can see, then, that the fact that a number has at most two square roots is by no means trivial.

Exercises 1-8

1. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Calculate AB, BA, (AB)A, (BA)A, (BA)B, B(BA), A(AB), ((BA)A)B, and ((AB)A)B.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Calculate:

(a) AB,
(b) BA,
(c) (AB)A,
(d) (BA)A,
(e) (BA)B,
(f) B(BA),
(g) A(AB),
(h) ((BA)A)B.
(i) ((AB)A)B.

3. Let A and B be as in Exercise 2, and let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Calculate AI, IA, BI, IB, and (AI)B.

4. Let

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

Show by computation that

(a) $(A + B)(A + B) \neq A^2 + 2AB + B^2$,
(b) $(A + B)(A - B) \neq A^2 - B^2$,

where A^2 and B^2 denote AA and BB, respectively.

5. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Calculate A^2, A^3, B^2, B^3, AB^2, A^2B.

6. Find at least 8 cube roots of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

7. Show that the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

satisfies the equation $A^2 = 0$. How many 2×2 matrices satisfying this equation can you find?

8. Show that the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

satisfies the equation $A(AA) = 0$.

1-9. Properties of Matrix Multiplication (Concluded).

We have seen that two basic laws governing multiplication in the algebra of ordinary numbers break down when it comes to matrices. The commutative law and the cancellation law do not hold. At this point, you might fear a total collapse of all the other familiar laws. This is not the case. Aside from the two laws mentioned, and the fact that, as we shall see later, many matrices do not have multiplicative inverses (receiprocals), the other basic laws of ordinary algebra generally remain valid for matrices. The associative law holds for the multiplication of matrices and there are distributive laws that unite addition and multiplication.

A few examples will aid us in understanding the laws.

Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}.$$

Then

$$A(BC) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 0 & 7 \end{bmatrix},$$

and

$$(AB)C = \left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \right) \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 0 & 7 \end{bmatrix}.$$

Thus,

$$A(BC) = (AB)C.$$

Again,

$$A(B + C) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix},$$

and

$$\begin{aligned} AB + AC &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}, \end{aligned}$$

so that

$$A(B + C) = AB + AC. \tag{1}$$

Since multiplication is not commutative, we cannot conclude from Equation (1) that the distributive principle is valid with the factor A on the right-hand side of $B + C$. Having illustrated the left-hand distributive law, we now illustrate the right-hand distributive law with the following example: We have

$$\begin{aligned} (B + C)A &= \left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 6 & 2 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} BA + CA &= \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 6 & 2 \end{bmatrix}. \end{aligned}$$

Thus,

$$(B + C)A = BA + CA.$$

You might note, in passing, that, in the above examples,

$$A(B + C) \neq (B + C)A.$$

These properties of matrix multiplication can be expressed as theorems.

Theorem 1-5. If

$$A = \left[a_{ij}\right]_{m \times n}, \quad B = \left[b_{jk}\right]_{n \times p}, \quad \text{and} \quad C = \left[c_{kh}\right]_{p \times q},$$

then

$$(AB)C = A(BC).$$

Proof. (Optional.) We have

$$AB = \left[\left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)\right]_{m \times p},$$

$$(AB)C = \left[\sum_{k=1}^{p}\left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)c_{kh}\right]_{m \times q};$$

$$BC = \left[\left(\sum_{k=1}^{p} b_{jk}c_{kh}\right)\right]_{n \times q},$$

$$A(BC) = \left[\sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{p} b_{jk}c_{kh}\right)\right]_{m \times q}.$$

Since the order of summation in a finite sum is arbitrary, we know that

$$\left[\sum_{k=1}^{p}\left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)c_{kh}\right]_{m \times q} = \left[\sum_{j=1}^{n} a_{ij}\left(\sum_{k=1}^{p} b_{jk}c_{kh}\right)\right]_{m \times q}.$$

Hence,

$$(AB)C = A(BC).$$

Theorem 1-6. If

$$A = \left[a_{ij}\right]_{m\times n}, \quad B = \left[b_{jk}\right]_{n\times p}, \quad \text{and} \quad C = \left[c_{jk}\right]_{n\times p},$$

then $A(B + C) = AB + BC$.

__Proof__. (Optional.) We have

$$(B + C) = \left[b_{jk} + c_{jk}\right]_{n\times p},$$

$$\begin{aligned} A(B + C) &= \left[\sum_{j=1}^{n} a_{ij}\,(b_{jk} + c_{jk})\right]_{m\times p} \\ &= \left[\sum_{j=1}^{n} a_{ij}b_{jk} + \sum_{j=1}^{n} a_{ij}c_{jk}\right]_{m\times p} \\ &= \left[\sum_{j=1}^{n} a_{ij}b_{jk}\right]_{m\times p} + \left[\sum_{j=1}^{n} a_{ij}c_{jk}\right]_{m\times p} \\ &= AB + AC. \end{aligned}$$

Hence,

$$A(B + C) = AB + AC.$$

__Theorem 1-7__. If

$$B = \left[b_{jk}\right]_{n\times p}, \quad C = \left[c_{jk}\right]_{n\times p}, \quad \text{and} \quad A = \left[a_{ki}\right]_{p\times q},$$

then $(B + C)A = BA + CA$.

__Proof__. The proof is similar to that of Theorem 1-6.

It should be noted that if the commutative law held for matrices, it would be unnecessary to prove Theorems 1-6 and 1-7 separately, since the two statements

$$A(B + C) = AB + AC$$

and

$$(B + C)A = BA + CA$$

would say exactly the same thing. For matrices, however, the two statements say different things, even though both are true. The order of factors is most important, since statements like

$$A(B + C) = AB + CA$$

and

$$(B + C)A = AB + CA$$

can be false for matrices.

Earlier we defined the zero matrix of order $m \times n$ and showed that it is the identity element for addition:

$$A + \underline{0} = A,$$

where A is any matrix of order $m \times n$. This zero matrix plays the same role in multiplication of matrices as the number zero does in the multiplication of real numbers. For example, we have

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2.$$

Theorem 1-8. For any matrix

$$A_{n \times p} = \left[a_{ij}\right]_{n \times p},$$

we have

$$0_{m \times n} A_{n \times p} = 0_{m \times p} \quad \text{and} \quad A_{n \times p} 0_{p \times q} = 0_{n \times q}.$$

The proof is easy and is left to the student.

Now we may be wondering if there is an identity element for the multiplication of matrices, namely a matrix that plays the same role as the number 1 does in the multiplication of real numbers. (For all real numbers a, $1a = a = a1$.) There _is_ such a matrix, called the unit matrix and denoted by the symbol I. The matrix I_2, namely,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

is called the unit matrix of order 2. The matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is called the unit matrix of order 3. In general, the _unit matrix_ of order n is the square matrix $\begin{bmatrix} e_{ij} \end{bmatrix}_{n \times n}$ such that $e_{ij} = 1$ for all $i = j$ and $e_{ij} = 0$ for all $i \neq j$ $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,n)$. We now state the important property of the unit matrix as a theorem.

Theorem 1-8. If A is an $m \times n$ matrix, then $AI_n = A$ and $I_m A = A$.

Proof. By definition, the entry in the i-th row and j-th column of the product AI_n is the sum $a_{i1}e_{1j} + a_{i2}e_{2j} + \cdots + a_{in}e_{nj}$. Since $e_{kj} = 0$ whenever $k \neq j$, all terms but one in this sum are zero and drop out. We are left with $a_{ij}e_{jj} = a_{ij}$. Thus the entry in the i-th row and j-th column of the product is the same as the corresponding entry in A. Hence $AI_n = A$. The equality $I_m A = A$ may be proved the same way. In most situations, it is not necessary to specify the order of the unit matrix since the order is inferred from the context. Thus, for

$$I_m A = A = AI_n,$$

we write

$$IA = A = AI.$$

For example, we have

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Exercises 1-9

1. Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Test the formulas

$$A(B + C) = AB + AC,$$

$$(B + C)A = BA + CA,$$

$$A(B + C) = AB + CA,$$

$$A(B + C) = BA + CA.$$

Which are correct, and which are false?

2. Let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Show that $AB \neq 0$, but $BA = 0$.

3. Show that for all matrices A and B of the form

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix},$$

we have

$$AB = BA.$$

Illustrate by assigning numerical values to a, b, c, and d, with $a,b,c,$ and d integers.

4. Find the value of x for which the following product is I:

$$\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -x & -14x & 7x \\ 0 & 1 & 0 \\ x & 4x & -2x \end{bmatrix}.$$

5. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}.$$

Show that $AB = BA$, that $AC = CA$, and that $BC = CB$.

6. Show that for all matrices A of the form

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix},$$

we have

$$AA = 0.$$

Illustrate by assigning numerical values to a and b.

7. Let

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Compute the following:

(a) DE,

(b) DF,

(c) EF,

(d) ED,

(e) FD,

(f) FE.

If $AB = -BA$, A and B are said to be anticommutative. What conclusions can be drawn concerning D, E, and F?

8. Show that the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ is a solution of the equation $AA - 5A + 7I = 0$.

9. Explain why, in matrix algebra,

$$(A + B)(A - B) \neq A^2 - B^2$$

except in special cases. Can you devise two matrices A and B that will illustrate the inequality? Can you devise two matrices A and B that will illustrate the special case? (Hint: Use square matrices of order 2.)

10. Show that if V and W are $n \times 1$ column vectors, then

$$V^t W = W^t V.$$

11. Prove that $(AB)^t = B^t A^t$, assuming that A and B are conformable for multiplication.

12. Using $\sum$ notation, prove the right-hand distributive law.

1-10. Fields and Rings

In this introductory chapter on matrices, we have defined several operations such as addition and multiplication. These operations differ from those of elementary algebra in that they cannot always be performed —

thus, we do not add a 2×2 matrix to a 3×3 matrix. Again, though a 4×3 matrix and a 3×4 matrix can be multiplied together, the product is neither 4×3 nor 3×4.

Let us fix our attention on the set of all 2×2 matrices, which we shall denote by M. Thus any 2×2 matrix is a member of M. Within this system, we can always add and multiply, and the sum and product of two elements of M are also elements of M; we express these facts by saying that M is closed with respect to addition and with respect to multiplication.

The results of this chapter will be used in Chapter 2 to check systematically that the set M of 2×2 matrices has the following properties:

The set is closed under addition.

Addition is commutative.

Addition is associative.

There exists an identity element for addition (the zero matrix).

There exists an additive inverse for each element in the set.

. . .

The set is closed under multiplication.

Multiplication is associative.

. . .

Multiplication is distributive over addition.

We have noticed that this algebra is different in two important aspects from the algebra of real numbers; namely, the commutative law for multiplication and the cancellation law do not hold.

There is a third significant difference that we shall explore more fully in later chapters but shall introduce now. Recall that the operation of subtraction was closely associated with that of addition. In order to solve equations of the form

$$A + X = B,$$

it is necessary to define the additive inverse or negative, $-A$. Then we have

$$
\begin{aligned}
A + X + (-A) &= B + (-A),\\
X + A + (-A) &= B + (-A),\\
X + \underline{0} &= B - A,\\
X &= B - A.
\end{aligned}
$$

Now "division" is closely associated with multiplication in a parallel manner. In order to solve equations of the form

$$AX = B,$$

it is necessary to define the multiplicative inverse (or reciprocal), which is denoted by the symbol A^{-1}. The defining property is $A^{-1}A = I = AA^{-1}$. This enables us to solve equations of the form

$$AX = B.$$

Thus

$$
\begin{aligned}
A^{-1}(AX) &= A^{-1}B,\\
(A^{-1}A)X &= A^{-1}B,\\
X &= A^{-1}B.
\end{aligned}
$$

Many matrices, other than the zero matrix $\underline{0}$, do not possess inverses; for instance,

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix}$$

are matrices of this sort, as we shall see in Chapter 2. This fact constitutes a very significant difference between the algebra of matrices and the algebra of real numbers.

The algebra of real numbers has the following properties:

The set is closed under addition.

Addition is commutative.

Addition is associative.

There exists an identity element for addition (zero).

There exists an additive inverse for each element in the set.

. . .

The set is closed under multiplication.

Multiplication is commutative.

Multiplication is associative.

There exists an identity element for multiplication (one).

There exists a multiplicative inverse for each element in the set (zero excepted).

. . .

Multiplication is distributive over addition.

Mathematical systems having the foregoing properties either of 2×2 matrices or of real numbers are sufficiently important, and are numerous enough, to be given special names. A set subject to two operations, called addition and multiplication, and possessing the properties listed for real numbers, is called a field. A set subject to two such operations and possessing all the properties listed for matrices is called a ring.

Since the list of defining properties for a field contains all the defining properties for a ring, it follows that every ring is a field. The set of 2×2 matrices has one more of the field properties; namely, there is an identity element I for multiplication. Accordingly we say that this set is a ring with an identity element.

We shall meet other such algebraic systems in Chapter 2; and in Chapters 4 and 5 we shall dwell on the additional fact that, as noted above in

Section 1.6, matrices can be multiplied by numbers. In fact, in Chapters 4 and 5 we shall see that vectors—which are matrices of a special sort—not only constitute interesting algebraic systems; they also have valuable physical applications.

Chapter 2

THE ALGEBRA OF 2 × 2 MATRICES

2–1. Introduction

In Chapter 1, we considered the elementary operations of addition and multiplication for rectangular matrices. This algebra is similar in many respects to the algebra of real numbers, although there are important differences. Specifically, we noted that the commutative law and the cancellation law do not hold in matrix algebra, and that division is not always possible.

With matrices, the whole problem of division is a very complex one; it is centered around the existence of a multiplicative inverse. You will recall that subtraction arose when we were solving the equation $A + X = B$ for the unknown matrix X. We needed a matrix $-A$, which is called the additive inverse for A, such that $A + (-A) = \underline{0}$. A similar pattern develops if we consider the problem of solving $AX = C$ for the unknown matrix X. This statement is misleading, although it seems innocuous. Let us ask a question: If you were given the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 8 & 9 & 0 & -1 \\ 4 & 5 & 6 & 5 \\ 0 & 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & \cdot\cdot & X_{14} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ X_{41} & \cdot\cdot & X_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix},$$

could you solve it for the unknown 4×4 matrix X? Do not be dismayed if your answer is "No." Eventually, we shall learn methods of solving this equation. However, the problem is complex and lengthy. In order to understand this problem in depth and, at the same time, comprehend the full significance of the algebra we have developed so far, we shall largely confine our attention in this chapter to a special subset of the set of all rectangular matrices; namely, we shall consider the set of 2×2 square matrices.

When one stands back and takes a broad view of the many different kinds of numbers that have been studied, one sees recurring patterns. For instance, let us look at the rational numbers for a moment. Here is a set of numbers that we can add and multiply. Under addition and multiplication, the set satisfies the following postulates:

The set is closed under addition.

Addition is commutative.

Addition is associative.

There exists an identity element for addition.

There exists an inverse element for each element under addition.

. . .

The set is closed under multiplication.

Multiplication is commutative.

Multiplication is associative.

There exists an identity element for multiplication.

There exists an inverse element for each element, except 0, under multiplication.

. . .

Multiplication is distributive over addition.

Since there exists a rational multiplicative inverse for each rational number except 0, division (except by 0) is always possible in the algebra of rational numbers. In other words, all equations of the form

$$ax = b,$$

where a and b are rational numbers and $a \neq 0$, can be solved for x in the algebra of rational numbers. For example: if we are given the equation

$$-\frac{2}{3}x = \frac{1}{2},$$

we multiply both sides of the equation by $-3/2$, the multiplicative inverse of $-2/3$. Thus we obtain

$$\left(-\frac{3}{2}\right)\left(-\frac{2}{3}\right)x = -\frac{3}{2}\cdot\frac{1}{2},$$

or

$$x = -\frac{3}{4},$$

which is a rational number.

The foregoing set of postulates is satisfied also by the set of real numbers, as we have noted previously on page 58. Any set that satisfies such a set of postulates is called a field. Thus both the set of real numbers and the set of rationals, which is a subset of the set of real numbers, are fields under addition and multiplication. There are many systems that have this same pattern. In each of these systems, division (except by 0) is always possible.

Now our immediate concern is to explore the problem of division in the set of matrices. There is no blanket answer that can readily be reached, although there is an answer that we can find by proceeding stepwise. At first, let us limit our discussion to the set of 2×2 matrices. We do this not only to consider division in a smaller domain, but also to study in detail the algebra associated with this subset. A more general problem of matrix division will be considered in Chapter 3.

Exercises 2-1

1. Determine which of the following sets are closed under the stated operation:

    (a) the set of integers under addition,

    (b) the set of even numbers under multiplication,

(c) the set $\{1\}$ under multiplication,

(d) the set of rational numbers under division,

(e) the set of positive rational numbers under division,

(f) the set of integers under the operation of squaring,

(g) the set of numbers $A = \{x \mid x \geq 3\}$ under addition.

2. Determine which of the following statements are true, and state which of the indicated operations are commutative:

(a) $2 - 3 = 3 - 2$,

(b) $4 \div 2 = 2 \div 4$,

(c) $3 + 2 = 2 + 3$,

(d) $\sqrt{a} + \sqrt{b} = \sqrt{b} + \sqrt{a}$, a and b positive,

(e) $a - b = b - a$, a and b real,

(f) $pq = qp$, p and q real,

(g) $\sqrt{-1} + 2 = 2 + \sqrt{-1}$.

3. Determine which of the following operations $\ddagger$, defined for positive integers in terms of addition and multiplication, are commutative:

(a) $x \ddagger y = x + 2y$ (for example, $2 \ddagger 3 = 2 + 6 = 8$),

(b) $x \ddagger y = 2xy$,

(c) $x \ddagger y = 2x + 2y$,

(d) $x \ddagger y = xy^2$,

(e) $x \ddagger y = x^y$,

(f) $x \ddagger y = x + y + 1$.

4. Determine which of the following operations $*$, defined for positive integers in terms of addition and multiplication, are associative:

(a) $x * y = x + 2y$ (for example, $(2 * 3) * 4 = 8 * 4 = 16$),

(b) $x * y = x + y$,

(c) $x * y = xy^2$,

(d) $x * y = x$,

(e) $x * y = \sqrt{xy}$,

(f) $x * y = xy + 1$.

5. Determine if the operation $*$ is distributive over the operation $\mp$, where the operations $\mp$ and $*$ are defined for positive integers in terms of addition and multiplication of real numbers:

   (a) $x \mp y = x + y$, $\quad x * y = xy$;

   (b) $x \mp y = 2x + 2y$, $\quad x * y = \frac{1}{2} xy$;

   (c) $x \mp y = x + y + 1$, $\quad x * y = xy$.

   Why is the answer the same in each case for left-hand distribution as it is for right-hand distribution?

6. In each of the following examples, determine if the specified set, under addition and multiplication, constitutes a field:

   (a) the set of all positive numbers,

   (b) the set of all rational numbers,

   (c) the set of all real numbers of the form $a + b\sqrt{2}$, where a and b are integers,

   (d) the set of all complex numbers of the form $a + bi$, where a and b are integers and $i = \sqrt{-1}$.

2-2. The Ring of 2 × 2 Matrices

Since we are confining our attention to the subset of 2 × 2 matrices, it is very convenient to have a symbol for this subset. We let M denote the set of all 2 × 2 square matrices. If A is a member, or element, of this set, we express this membership symbolically by $A \in M$.

Since all elements of M are matrices, our general definitions of addition and multiplication prevail over this subset. For example, we have

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix};$$

also,

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 7 & 1 \end{bmatrix}.$$

For convenience of reference, let us repeat the defining postulates for a ring, which we listed in the last section of Chapter 1. A ring is a set that possesses the following properties under addition and multiplication:

The set is closed under addition.

Addition is commutative.

Addition is associative.

There is an identity element for addition.

There is an additive inverse for each element.

. . .

The set is closed under multiplication.

Multiplication is associative.

. . .

Multiplication is distributive over addition.

Does the set M satisfy these properties? It seems clear that it does, but the answer is not quite obvious. Consider the set of all real numbers. This set is a field because there exists, among other things, an additive inverse for each number in this set. Now the positive integers are a subset of the real numbers. Does this subset contain an additive inverse for each element? Since we do not have negative integers in the set under consideration, the answer is "No"; therefore, the set of positive integers is not a field. Clearly, a subset does not necessarily have the same properties as the complete set.

To be certain that the set M is a ring, we must systematically make sure that each criterion is satisfied. For the most part, our proof will be a reiteration of the material in Chapter 1, since the general properties of matrices will be valid for the subset M of 2×2 matrices. The sum of two 2×2 matrices is a 2×2 matrix; thus, the set is closed under addition. The general proofs of commutativity and associativity are valid. The unit matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ;$$

the zero matrix is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} ;$$

and the additive inverse of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} .$$

When we consider the multiplication of 2×2 matrices, we must first verify that the product is an element of this set, namely a 2×2 matrix. Recall that the number of rows in the product is equal to the number of rows in the left-hand factor, and the number of columns is equal to the number of columns in the right-hand factor. Thus, the product of two elements of the set M must be an element of this set, namely a 2×2 matrix; accordingly, the set is closed under multiplication. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} .$$

The general proof of associativity is valid for elements of M, since it is for rectangular matrices. Also, both of the distributive laws hold for elements of M by the same reasoning. For example,

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} ,$$

and also

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} ;$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix}$$

and also

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix} .$$

Since we have demonstrated that each of the ring postulates is fulfilled, we have proved that the set M of 2×2 matrices is a ring under addition and multiplication. We state this result formally as a theorem.

Theorem 2-1. The set M of 2×2 matrices is a ring under addition and multiplication.

Furthermore, we know that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity element for multiplication. Thus the set M is a ring with an identity element.

At this time, we should verify that the commutative law for multiplication and the cancellation law are not valid by giving counterexamples. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix},$$

but

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix},$$

so that the commutative law for multiplication does not hold. Also,

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so that the cancellation law does not hold.

Exercises 2-2

1. Determine if the set of all integers is a ring under the operations of addition and multiplication.

2. Determine which of the following sets are rings under addition and multiplication:

   (a) the set of numbers of the form $a + b\sqrt{2}$, where a and b are integers;

   (b) the set of four fourth roots of unity, namely, $+1$, -1, i and $-i$;

   (c) the set of numbers $a/2$, where a is an integer.

3. Determine if the set of all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, with $a \in R$, forms a ring under addition and multiplication as defined for matrices.

4. Determine if the set of all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & a^2 \end{bmatrix}$, with $a \in R$, forms a ring under addition and multiplication as defined for matrices.

2-3. The Uniqueness of the Multiplicative Inverse

Once again we turn our attention to the problem of matrix division. As we have seen, this problem arises when we seek to solve a matrix equation of the form

$$AX = C.$$

Let us look at a parallel equation concerning real numbers,

$$ax = c.$$

Each nonzero number a has a reciprocal $1/a$, which is often designated a^{-1} and whose defining property is $aa^{-1} = 1$. Since multiplication of real numbers is commutative, it follows that $a^{-1}a = 1$. Hence if a is a nonzero number, then there is a number b, called the multiplicative inverse of a, such that

$$ab = 1 = ba \qquad (b = a^{-1}).$$

Given an equation $ax = c$, b enables us to find a solution for x; thus,

$$axb = cb,$$
$$abx = cb,$$
$$1x = cb,$$
$$x = cb.$$

Now our question concerning division by matrices can be put in another way. If $A \in M$, is there a $B \in M$ for which the equation

$$AB = I = BA$$

is satisfied? We shall employ the more suggestive notation A^{-1} for the inverse, so that our question can be restated: Is there an element $A^{-1} \in M$ for which the equation

$$AA^{-1} = I = A^{-1}A$$

is satisfied? Since we shall often be using this defining property, let us state it formally as a definition.

Definition 2-1. If $A \in M$, then an element A^{-1} of M is an inverse of A provided

$$AA^{-1} = I = A^{-1}A.$$

If there were an element B corresponding to each element $A \in M$ such that

$$BA = I = BA,$$

then we could solve all equations of the form

$$AX = C,$$

since we would have

$$\begin{aligned} B(AX) &= BC, \\ (BA)X &= BC, \\ IX &= BC, \\ X &= BC, \end{aligned}$$

and clearly this value satisfies the original equation.

From the fact that there is a multiplicative inverse for every real number except zero, we might wrongly infer a parallel conclusion for matrices. As stated in Chapter 1, not all matrices have inverses. Our knowledge that 0 has no inverse suggests that the zero matrix $\underline{0}$ has no inverse. This is true, since

$$\underline{0}X = \underline{0}$$

for all $X \in M$, so that there cannot be any $X \in M$ such that

$$\underline{O}X = I.$$

But there is a more fundamental difficulty than this. Let us take the nonzero matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and try to solve the equation

$$AX = I, \qquad \text{for } X \in M.$$

If we let

$$X = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

then

$$AX = \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix}.$$

Hence, no matter what entries we take for X, we cannot have

$$AX = I$$

since the entry in the lower right-hand corner of AX is zero, and the entry in the lower right-hand corner of I is 1.

At this point, you might be thinking that no matrix has an inverse. Do not move too fast! Note that

$$I \cdot I = I = I \cdot I.$$

This means that I is its own inverse, just as 1 is its own inverse among the numbers.

Also, let us note that

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} .$$

Thus the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

has the inverse

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} .$$

Consequently, the equation

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

may be solved by multiplying both sides by A^{-1}, thus:

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} X = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} ,$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1/2 & 1 \\ 3/2 & 2 \end{bmatrix} ,$$

$$X = \begin{bmatrix} 1/2 & 1 \\ 3/2 & 2 \end{bmatrix} .$$

This is a specific illustration of a general pattern. Let a be any nonzero number. Now

$$\begin{aligned} I &= 1I \\ &= aa^{-1} I \\ &= aa^{-1} II \qquad \text{(since } I = II\text{)}. \end{aligned}$$

Since the multiplication of real numbers and of matrices is associative and commutative, it follows that for all real numbers a and b, and all 2 × 2 matrices X and Y, we have

$$abXY = (aX)(bY).$$

In particular, then,

$$I = (aI)(a^{-1}I).$$

Since $aa^{-1} = a^{-1}a$, we can also state that

$$I = (a^{-1}I)(aI).$$

This result enables us to enumerate a large number of matrices and their inverses. Thus, let $A = aI$; then $A^{-1} = a^{-1}I$. For example, if $a = 3$ then

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}.$$

If $a = 0.2$, then

$$A = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}.$$

At least we know that there are a great many matrices A with the property that there is a corresponding matrix B such that

$$AB = I = BA.$$

Before turning to the problem of finding those matrices that have inverses, let us show first that if a matrix has an inverse, it has only one inverse; that is, this inverse is unique. For instance, in the example directly above, we saw that

$$A^{-1} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{if} \quad A = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

We wish to show that there is no other inverse. Suppose that we have elements A, B, and C of M such that

$$AB = I = BA,$$

and

$$AC = I = CA;$$

that is, A has an inverse B and A also has an inverse C. Multiply the first of these two equations on the left by C. Then

$$C(AB) = CI,$$

or

$$(CA)B = C$$

since multiplication is associative and I is the unit matrix. But now $CA = I$. Hence

$$IB = C,$$

or

$$B = C.$$

This result is so important that we call it a theorem and state it formally:

Theorem 2-2. If $A \in M$ and if there exists A^{-1}, $A^{-1} \in M$, such that

$$AA^{-1} = I = A^{-1}A,$$

then A^{-1} is unique; that is, there is no other solution X of the equations

$$AX = I = XA.$$

Now we can readily show that A is the inverse of A^{-1} if we know that A^{-1} is the inverse of A. This may seem a bit trivial, but it is important enough to state formally and prove.

Theorem 2-3. If $A \in M$ and if A has an inverse A^{-1}, then A^{-1} also has an inverse; namely, A is the inverse of A^{-1}.

Proof. Since A^{-1} is the inverse of A, this means, by definition, that

$$AA^{-1} = I = A^{-1}A.$$

However, the statement of equality can be given in reverse order:

$$A^{-1}A = I = AA^{-1}.$$

This, by definition, is the statement that A is the inverse of A^{-1}.

Exercises 2–3

1. Show that each of the following matrices does not have an inverse:

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, (d) $\begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix}$.

2. Which of the following pairs of elements of M are inverses of one another?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$ and $\begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$,

(c) $\begin{bmatrix} 2 & 4 \\ 6 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -4 \\ -6 & 2 \end{bmatrix}$,

(d) $\begin{bmatrix} -5 & 7 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

(e) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

3. Use the argument in the text to show that, since

$$\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \underline{0},$$

neither of the matrices in the product is invertible (has an inverse).

4. Show that if $a^2 + bc = 0$, then

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}^2 = \underline{0},$$

and hence that

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

has no inverse.

5. Show that if $A \in M$, $B \in M$, $B \neq \underline{0}$, and $AB = \underline{0}$, then A cannot have an inverse. Can B have an inverse?

6. Show that if $A \in M$, and $A^2 - 4A = \underline{0}$, then either $A = 4I$ or A has no inverse. (Hint: Factor the left-hand side and note Exercise 5.)

7. Show that if $A \in M$, $B \in M$, $C \in M$, and $AB = I = CA$, then $B = C$.

8. Show by direct computation that

$$\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

9. The matrices

$$\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$$

are inverses of one another. Are their squares also inverses? Their transposes?

10. Since

$$A^2 = A \cdot A,$$
$$A^3 = A \cdot A^2,$$
$$A^4 = A \cdot A^3,$$

we can readily demonstrate that A^{n-1} is the inverse of A if $A^n = I$. Using this information, compute the inverse of each of the following matrices:

(a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

(b) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$,

(c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

11. Let

$$B = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

and compute B^2 and B^3 if $0 = 120^\circ$.

12. If

$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix},$$

verify that

$$A^2 - 2A + I = \underline{0}.$$

Does the transpose of A also satisfy this same equation?

13. Prove that if $A \epsilon M$, if p, q, and r are numbers, and if

$$pA^2 + qA + rI = \underline{0}$$

with $r \neq 0$, then A has an inverse. (Hint: Transpose the "constant term" and factor the remaining terms. Be careful of what happens if p or q is 0.)

14. Prove, by direct substitution, that if

$$X = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

then

$$X^2 - (p + s)X + (ps - qr)\,I = \underline{0}.$$

Show that X has an inverse if and only if $ps - qr \neq 0$. (Hint: Use Exercise 13.)

15. Use the result of Exercise 14 to show that if $X^2 = \underline{0}$ then $ps - qr = 0$ and $p + q = 0$. (Perhaps you may have to consider several cases in the proof.)

2-4. The Inverse of a Matrix of Order 2

At this point, we have proved that the inverse of a 2 × 2 matrix, if it exists, is unique. Also, we know that there are some matrices that have inverses and there are some that do not have inverses. But we have not yet developed any general methods of attacking the problem. Certainly our algebra will lack power unless general methods are developed. We are in a situation similar to that in which a student finds himself when he has learned to factor a quadratic equation and has not yet learned the quadratic formula. He can find the roots of many quadratic equations by trial, but he has no means for solving all these equations.

It is our purpose now to develop a general method of determining the inverse of a 2 × 2 matrix when it exists. We shall begin with a matrix whose entries are specific numbers and then duplicate our procedure with a matrix whose entries are more general. To start, we shall consider the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix}$$

and determine whether there is an inverse B such that $AB = I = BA$. If we let

$$B = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

then

$$\begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

or

$$\begin{bmatrix} 3p - r & 3q - s \\ 5p - 2r & 5q - 2s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since these two matrices are equal, the individual entries are equal. Thus we have four equations,

$$3p - 4 = 1, \quad (1) \qquad 3q - s = 0, \quad (3)$$

$$5p - 2r = 0, \quad (2) \qquad 5q - 2s = 1. \quad (4)$$

After multiplying Equation (1) by 2, we subtract Equation (2) from Equation (1) and obtain

$$p = 2.$$

By substituting this value of p in either Equation (1) or Equation (2), we obtain

$$r = 5.$$

Equations (3) and (4) can be solved similarly, yielding

$$q = -1 \quad \text{and} \quad s = -3.$$

Now if we substitute these values for p, q, r, and s, we obtain

$$B = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}.$$

To demonstrate that B is the inverse of A, we must show that $AB = I = BA$. This is easy:

$$\begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}.$$

Using the notation for the inverse of a matrix introduced earlier, we may write

$$\begin{bmatrix} 3 & -5 \\ 5 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}.$$

In our next step, we shall follow the same pattern as above; but now we shall use a general notation for our matrix A. Instead of having specific real numbers for entries, we let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

As before, we represent the inverse, if it exists, as

$$B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Assuming $AB = I$, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This matrix equation may be written as four equations,

$$ap + br = 1, \quad (5) \qquad aq + bs = 0, \quad (7)$$

$$cp + dr = 0, \quad (6) \qquad cq + ds = 1. \quad (8)$$

Since we wish to find values for p, q, r, and s, in terms of the real numbers a, b, c, and d, we multiply Equation (5) by d, Equation (6) by b, and then subtract. We obtain

$$adb - bcp = d,$$

or

$$(ad - bc)p = d.$$

Repeating this process, using appropriate pairs of equations, we obtain

$$(ad - bc)q = -b, \quad (ad - bc)r = c, \quad (ad - bc)s = a.$$

Should it happens that $ad - bc = 0$, then it follows from the four equations, above, that $a = b = c = d = 0$, so that $A = \underline{0}$.

We have seen in Section 2.3 that the zero matrix does not have an inverse. Hence if $ad - bc = 0$ we have a contradiction of the assumption that the matrix A has an inverse B. In other words, if A has an inverse, then $ad - bc \neq 0$.

Temporarily, let us denote the number $ad - bc$ by h. Now if $h \neq 0$, we may write

$$p = \frac{d}{h}, \quad r = -\frac{c}{h}, \quad q = -\frac{b}{h}, \quad s = \frac{a}{h}.$$

Substituting these values appropriately in B, we obtain

$$B = \begin{bmatrix} \frac{d}{h} & -\frac{b}{h} \\ -\frac{c}{h} & \frac{a}{h} \end{bmatrix} = \frac{1}{h}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

In order to show that this matrix is the inverse of A, we check:

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{h} & -\frac{b}{h} \\ -\frac{c}{h} & \frac{a}{h} \end{bmatrix} = \begin{bmatrix} \frac{ad-bc}{h} & \frac{-ab+ab}{h} \\ \frac{cd-cd}{h} & \frac{-cb+ad}{h} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

We must also make sure that $BA = I$, thus:

$$BA = \begin{bmatrix} \frac{d}{h} & -\frac{b}{h} \\ -\frac{c}{h} & \frac{a}{h} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{ad-bc}{h} & \frac{db-bd}{h} \\ \frac{-ca+ac}{h} & \frac{-bc+ad}{h} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

The fact that the relationship $BA = I$ follows from the relationship $AB = I$ is quite significant. While the definition of the inverse demands the existence and equality of what are called left and right inverses, we have shown that for

2×2 matrices the existence of one implies the existence of the other and that if they exist then they are, in fact, the same. Since the multiplication of matrices is not generally commutative, we might have expected otherwise.

We shall state our result formally as a theorem.

Theorem 2-4. If the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse, then $h = ad - bc \neq 0$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{h} & -\frac{b}{h} \\ -\frac{c}{h} & \frac{a}{h} \end{bmatrix}$$

Also, we state the converse of this result concerning h:

Theorem 2-5. If $h = ad - bc \neq 0$, then the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse, which is

$$\begin{bmatrix} \frac{d}{h} & -\frac{b}{h} \\ -\frac{c}{h} & \frac{a}{h} \end{bmatrix}.$$

Proof. Direct multiplication shows that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{h} & -\frac{b}{h} \\ -\frac{c}{h} & \frac{a}{h} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{d}{h} & -\frac{b}{h} \\ -\frac{c}{h} & \frac{a}{h} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Exercises 2-4

1. For each of the following matrices, determine whether the inverse exists; if it does exist, find it:

(a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, (e) $\begin{bmatrix} -3 & 7 \\ 9 & 21 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, (f) $\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$,

(c) $\begin{bmatrix} -2 & 0 \\ 3 & 4 \end{bmatrix}$, (g) $\begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$.

(d) $\begin{bmatrix} 2 & a \\ 0 & -7 \end{bmatrix}$,

2. Each of the following matrices is actually a function in the sense that it depends on the value assigned to x, where $x \in R$. Determine those values of x for which the matrix has no inverse.

(a) $\begin{bmatrix} x^2 & 1 \\ 1 & x \end{bmatrix}$, (c) $\begin{bmatrix} x+2 & 0 \\ x^4 & x-1 \end{bmatrix}$,

(b) $\begin{bmatrix} x^3 & x \\ 0 & 1 \end{bmatrix}$, (d) $\begin{bmatrix} x^2 & x-1 \\ 2 & 3 \end{bmatrix}$.

3. Show that each matrix of the form

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

has an inverse and find it. Show that the product of two such matrices (different values of θ) is again such a matrix. (Hint: Use the addition formulas from trigonometry.)

4. Show that if $A \in M$ then A has an inverse if and only if its transpose has an inverse. If A has an inverse show that

$$\text{transpose } (A^{-1}) = (\text{transpose } A)^{-1}.$$

5. Prove Theorem 2-3 by first computing A^{-1} by Theorem 2-5 and then using Theorem 2-5 again to compute the inverse of A^{-1}.

6. Under the assumption that the element A of M has an inverse, show how to solve the equation $AX = B$, with $B \in M$. Apply this to solve the

following equations:

(a) $2x + 3z = 9,$
$\quad -x + 4z = 10;$

(b) $3x + z = 0;$
$\quad -2x + z = 1;$

(c) $2y + 3w = 0,$
$\quad -y + 4w = 0;$

(d) $3y + w = 1,$
$\quad -2y + w = 0.$

2-5. The Determinant Function

We have seen that the criterion for the existence of an inverse for the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

involves the value of the expression $ad - bc$. If $ad - bc \neq 0$, the inverse does exist; if $ad - bc = 0$, the inverse does not exist. Each 2×2 matrix determines one value for $ad - bc$. For example,

$$\text{if } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{then } ad - bc = 1(1) - 0(0) = 1;$$

$$\text{if } A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, \quad \text{then } ad - bc = 2(6) - 3(4) = 0;$$

$$\text{if } A = \begin{bmatrix} 0.5 & 3 \\ 4 & 0.6 \end{bmatrix}, \quad \text{then } ad-bc = 0.5(0.6) - 3(4) = 11.7.$$

(Note that the second matrix does not have an inverse.) With each matrix there is thus associated one value, a real number determined by the entries. It is convenient to give a name to this number, the value of the expression $ad - bc$, which is associated with the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Definition 2-2. If

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then $\delta(X) = ad - bc$ is called the determinant of X.

Thus δ assigns to each member X of M a real number $\delta(X)$, read "delta of X." It is appropriate to regard this assignment or mapping as a function from the set of 2×2 matrices M to the set of real numbers R,

$$\delta : M \longrightarrow R.$$

The function δ has interesting properties, some of which we shall demonstrate.

First let us compute the values $\delta(X)$ for a few products:

(a) If

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix},$$

then

$$\delta(A) = 3(2) - 2(1) = 4,$$

$$\delta(B) = 0(1) - 3(2) = -6.$$

$$AB = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ 4 & 5 \end{bmatrix},$$

$$\delta(AB) = 4(5) - 11(4) = -24.$$

(b) If

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix},$$

then

$$\delta(A) = -1(3) - 2(0) = -3,$$

$$\delta(B) = 8(1) - 0(3) = 8,$$

$$AB = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 9 & 3 \end{bmatrix},$$

94

$$\delta(AB) = -2(3) - 2(9) = -24.$$

We might suspect that $\delta(AB) = \delta(A) \times \delta(B)$! This is true and we shall now prove it.

Theorem 2-6. If $A \in M$ and $B \in M$, then

$$\delta(AB) = \delta(A)\,\delta(B).$$

Proof. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix};$$

then

$$AB = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix},$$

$$\begin{aligned} \delta(AB) &= (ap + br)(cq + ds) - (aq + bs)(cp + dr) \\ &= apcq + apds + brcq + brds \\ &\qquad - aqcp - aqdr - bscp - bsdr \\ &= apds + brcq - aqdr - bscp, \qquad (1) \end{aligned}$$

$$\begin{aligned} \delta(A) &= ad - bc, \\ \delta(B) &= ps - qr, \\ \delta(A)\cdot\delta(B) &= (ad - bc)(ps - qr) \\ &= adps - adqr - bcps + bcqr. \qquad (2) \end{aligned}$$

By rearranging the terms in expressions (1) and (2), we see that

$$\delta(AB) = \delta(A)\cdot\delta(B), \qquad \text{q.e.d.}$$

Let us look at another example; let

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} .$$

Now if

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$X^{-1} = \frac{1}{\delta(X)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} .$$

Hence

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \quad \text{and} \quad B^{-1} \begin{bmatrix} -\frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix} .$$

Further,

$$\delta(A) = 3(2) - 2(1) = 4,$$

$$\delta(B) = 0(1) - 3(2) = -6,$$

$$\delta(A^{-1}) = \frac{1}{2}\ \frac{3}{4} - \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right) = \frac{1}{4} ,$$

$$\delta(B^{-1}) = -\frac{1}{6}(0) - \frac{1}{2}\left(\frac{1}{3}\right) = -\frac{1}{6} .$$

Theorem 2-7. If A is a 2×2 matrix, and A has a multiplicative inverse, then

$$\delta(A^{-1}) = \frac{1}{\delta(A)} .$$

Proof. We have

$$AA^{-1} = I,$$

$$\delta(AA^{-1}) = \delta(I).$$

But by computing $\delta(I)$, we see that

$$\delta(I) = 1,$$

whence

$$\delta(AA^{-1}) = 1,$$

so that

$$\delta(A)\cdot\delta(A^{-1}) = 1,$$

or

$$\delta(A^{-1}) = \frac{1}{\delta(A)}.$$

We are now in a position where we can prove quite a significant theorem, which will give us the power to decide when a product AB has an inverse and what the inverse is.

Theorem 2-8. If A and B are 2 × 2 matrices, and if A and B have inverses, then AB has an inverse $(AB)^{-1}$, namely,

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof. Recall that we have

$$\delta(A) \neq 0,$$

$$\delta(B) \neq 0,$$

$$\delta(A)\cdot\delta(B) = \delta(AB).$$

Hence,

$$\delta(AB) \neq 0,$$

which means that AB has an inverse, by Theorem 2-5. To complete the proof of our theorem, we need only exhibit a matrix X such that

$$ABX = I = XAB.$$

Let

$$X = B^{-1} A^{-1}.$$

Then

$$\begin{aligned} ABX &= AB\, B^{-1} A^{-1} \\ &= A(BB^{-1})\, A^{-1} \\ &= A(I)\, A^{-1} \\ &= AA^{-1} \\ &= I. \end{aligned}$$

Hence $B^{-1} A^{-1}$ is a right inverse. Similarly, we show that

$$B^{-1} A^{-1} AB = I.$$

Thus $B^{-1} A^{-1}$ is the inverse of AB. This completes the proof.

For example, let

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}.$$

Then

$$A^{-1} = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

Now

$$AB = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 7 & 19 \end{bmatrix},$$

whence

$$(AB)^{-1} = \begin{bmatrix} 1 & 3 \\ 7 & 19 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{19}{2} & \frac{3}{2} \\ \frac{7}{2} & \frac{1}{2} \end{bmatrix}.$$

But also,

$$B^{-1} A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{19}{2} & \frac{3}{2} \\ \frac{7}{2} & \frac{1}{2} \end{bmatrix}.$$

Thus, for our example we have $(AB)^{-1} = B^{-1} A^{-1}$.

There are many other theorems that can be developed from the concept of a determinant function. A few of these will be included in the exercises that follow. It is worth noting, though we shall not prove it, that there is a determinant function associated with the other sets of square matrices, that is, with those of order 1, 3, 4,..., and that similar theorems hold for them.

Exercises 2-5

1. Verify Theorem 2-8 for the matrices

   (a) $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix};$

   (b) $A = \begin{bmatrix} t^2 & 1 \\ -1 & t \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$

   (c) $A = \begin{bmatrix} x & x^2 \\ x^3 & x^4 \end{bmatrix}, \quad B = \begin{bmatrix} x & -x \\ 3 & 4 \end{bmatrix}.$

2. Show that

$$\delta(tA) = t^2 \, \delta(A)$$

   for any $A \in M$ and any $t \in R$.

3. Show that $\delta(A)$ is the constant term in the polynomial $\delta(A-tI)$.

4. If

$$A = \begin{bmatrix} x & 1 \\ x^2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix},$$

find $\delta(A)$ and $\delta(B^{-1}AB)$ and show that they are equal.

5. Show that if $A \in M$, $B \in M$, and B is invertible, then

$$\delta(B^{-1}AB) = \delta(A).$$

6. Show that if A^t is the transpose of A then

$$\delta(A) = \delta(A^t),$$

and conclude that

$$\delta(AA^t) \geq 0$$

for any $A \in M$.

7. The expression $\delta(A - tI)$ is a polynomial in t. For each of the following matrices A, expand this polynomial and find its zeros:

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$,

(b) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$,

(c) $\begin{bmatrix} t & 0 \\ -t & 1 \end{bmatrix}$,

(d) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

8. Let

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

and expand the polynomial $\delta(AA^t - xI)$. Is this the same as the polynomial $\delta(A^tA - xI)$? Are these two polynomials the same for every matrix $A \in M$?

2–6. The Group of Invertible Matrices

In this chapter, we have been restricting our attention to the set M of 2×2 matrices. This set is, itself, a subset of the set of all rectangular matrices. Now this set M can be separated into interesting subsets. In the preceding section, we have divided M into two complementary subsets, the set of 2×2 matrices that do not have inverses and the set of 2×2 matrices that do have inverses. In this section, we shall confine our attention principally to the set of invertible matrices. It is convenient to denote this set by the symbol M_i.

Let us summarize certain facts about the set M_i of invertible matrices:

(a) If $A \in M_i$, and $B \in M_i$, then $AB \in M_i$.

(b) If $A \in M_i$, $B \in M_i$, and $C \in M_i$, then $A(BC) = (AB)C$.

(c) In M_i, there is an identity element, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(d) If $A \in M_i$, then A has an inverse A^{-1}, and $A^{-1} \in M_i$.

Not only does the set M_i satisfy each of these conditions, but there are many subsets of M_i that satisfy conditions analogous to them. Any set S of matrices that satisfies conditions (a), (b), (c), and (d), with S in place of M_i, will be called a group. The concept of a group is fundamental and extremely important in mathematics. More generally, any set of elements $A, B, C, \ldots$, not necessarily matrices, satisfying the foregoing properties relative to an operation (not necessarily matrix multiplication), is defined to be a group. You will note that only one operation is involved in the group properties. Although we shall later introduce a few examples of the more general concept, for the moment let us consider some examples of groups of invertible matrices.

The smallest set of invertible matrices that constitutes a group is the set whose one element is the unit matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$! Since II = I, condition (a) is satisfied; and condition (b) is automatically fulfilled by any set of square matrices. Certainly I is a member of the set, so that condition (c) is satisfied. For condition (d), there must be an inverse for every element; but in our present set, the only element I is its own inverse.

All quite simple, isn't it? Was it obvious?

Another set that constitutes a group is the set $\{I, -I\}$. Again conditions (b) and (c) obviously are satisfied. Since

$$(I)(-I) = (-I)(I) = -I$$

and

$$(I)(I) = (-I)(-I) = I,$$

conditions (a) and (d) also are satisfied.

The third set that we shall show to be a group is a bit more significant. The set of all elements $A \in M$ such that $\delta(A) = 1$ is a group. The proof is a bit more difficult, and we must check carefully each one of the defining properties. To provide a language that will be helpful, let us denote this set by W, thus:

$$W = \{A \mid A \in M \text{ and } \delta(A) = 1\}.$$

Let us verify first that condition (a) is satisfied. If $A \in W$ and $B \in W$, then $\delta(A) = 1$ and $\delta(B) = 1$. Since $\delta(AB) = \delta(A)\ \delta(B)$ by Theorem 2-6, we have

$$\delta(AB) = \delta(A)\ \delta(B) = (1)(1) = 1,$$

and thus $AB \in W$.

Property (b) holds automatically.

For property (c), since $\delta(I) = 1$, it is clear that $I \in W$.

To demonstrate that condition (d) is satisfied, we must show not only that each element of W has an inverse but also that the inverse is an element of W. Now, if $A \in W$, then $\delta(A) = 1$. Since $\delta(A) \neq 0$, A has an inverse A^{-1}, by Theorem 2-5. Since

$$AA^{-1} = I$$

and

$$\delta(I) = 1,$$

we have

$$\begin{aligned} \delta(AA^{-1}) &= 1, \\ \delta(A)\,\delta(A^{-1}) &= 1, \\ 1\,\delta(A^{-1}) &= 1, \\ \delta(A^{-1}) &= 1. \end{aligned}$$

Hence $A^{-1} \in W$, and we have now demonstrated that W is a group.

In our last example, we shall discuss all matrices of the form

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \qquad (x,y = \text{real numbers})$$

and denote this set by Z, $Z \subset M$.

We observe first that the product of any two members of this set Z is also a member of Z. We have, indeed,

$$\begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 - y_1y_2 & x_1y_2 + y_1x_2 \\ -(x_2y_1 + x_1y_2) & -y_1y_2 + x_1x_2 \end{bmatrix}.$$

Condition (b) is automatically satisfied; and clearly I is a member of Z, so that condition (c) is satisfied.

In considering condition (d), we run into trouble. The zero matrix is an element of this set, but the zero matrix does not have an inverse. The set of all matrices of the form

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

does not form a group. Although the set Z does not satisfy the four conditions, a subset Z_1 of Z, defined by

$$Z_1 = \{A \mid A \in Z \text{ and } \delta(A) = 1\},$$

does satisfy the conditions and is therefore a group.

The demonstration is easy. Let $A \in Z_1$ and $B \in Z_1$. We know that $AB \in Z$, as already shown; and, since $\delta(A) = 1$ and $\delta(B) = 1$, we know that $\delta(AB) = 1$. Hence $AB \in Z_1$, and therefore condition (a) is satisfied. Obviously, condition (b) also is satisfied. We know that $I \in Z$ and that $\delta(I) = 1$; hence, $I \in Z_1$, so that condition (c) is satisfied. Finally, for condition (d), we must show that if $A \in Z_1$ then there is an inverse A^{-1} such that $A^{-1} \in Z_1$. We follow the pattern set in an earlier illustration. Since $\delta(A) = 1$, there is an inverse. Then, using the fact that $\delta(AA^{-1}) = \delta(I)$, we proceed to show that $\delta(A^{-1}) = 1$, which means that $A^{-1} \in Z_1$. Having demonstrated that the four groups postulates are satisfied, we conclude that we have a group.

Before considering the more general concept of a group, we shall demonstrate a fruitful correspondence between the elements of Z_1 and the points on a unit circle, which will let us examine the geometric meaning of Z_1.

If

$$A = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

is any element of Z_1, we have $\delta(A) = 1$; that is, we have

$$x^2 + y^2 = 1.$$

Now, if we let x and y be coordinates of a point (x,y), we are able to establish a one-to-one correspondence between the elements of Z_1 and the points on a unit circle:

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \longleftrightarrow (x, y).$$

The set of matrices is thus mapped onto the set of points in such a way that to each matrix there corresponds exactly one point of the set, and to each point of the set there corresponds exactly one matrix.

The point (x, y) is on the circle of radius 1 with center at the origin, as shown in Figure 2-1.

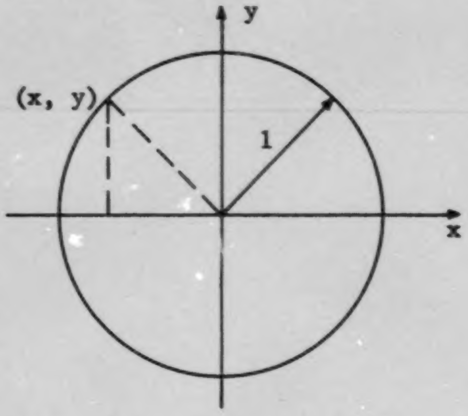


Figure 2-1. The unit circle.

Let us call this circle the unit circle and denote it by Q.

Thus

$$Q = \left\{ (x, y) \mid x \in R,\ \ y \in R,\ \text{ and }\ x^2 + y^2 = 1 \right\}$$

A geometrical meaning can be assigned to the inverse of any element of Z_1. If

$$A = \begin{bmatrix} x & y \\ -y & x \end{bmatrix},$$

then we can readily compute A^{-1} by Theorem 2-5, to obtain

$$A^{-1} = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

Recalling the one-to-one correspondence between the matrices of Z_1 and the points of Q (the unit circle),

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \longleftrightarrow (x, y),$$

we see, by examining Figure 2-2, that the correspondent of A^{-1} is the reflection in the x axis of the correspondent of A.

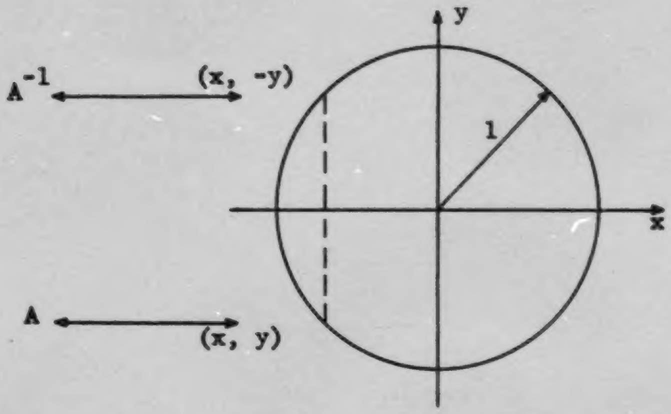


Figure 2-2. Geometric representation of inverse matrices A and $A^{-1} \in Z_1$.

Although a full discussion of the general notion of a group would be too extensive for this book, a few words are in order. The definition of an abstract group is stated somewhat differently from the defining properties given on page 93, although the abstract definition implies the latter.

Definition 2-3. A group is a set G of elements, $a, b, c, \ldots,$ on which a binary operation $\circ$ (read "circle") is defined, such that the following properties are satisfied:

(a) If $a \in G$ and $b \in G$, then $a \circ b \in G$. (Closure property.)

(b) If $a \in G$, $b \in G$, and $c \in G$, then $a \circ (b \circ c) = (a \circ b) \circ c$. (Associative property.)

(c) There exists a unique element i, $i \in G$, such that $i \circ a = a = a \circ i$ for all $a \in G$. (Identity property.)

(d) For each $a \in G$, there exists an element a^{-1}, $a^{-1} \in G$, such that $a^{-1} \circ a = a \circ a^{-1} = i$. (Inverse property.)

If, in addition, the following condition is fulfilled, the group is said to be commutative of abelian:

(e) For each $a \in G$ and each $b \in G$, $a \circ b = b \circ a$. (Commutative property.)

Although the operations we are most concerned with in mathematics are addition and multiplication, we are not restricted to these in the foregoing abstract definition. For instance, a very helpful exercise, not only for understanding the notion of a group but also for comprehending a finite number system, is the addition associated with a clock face; see Figure 2-3. This furnishes us with a group. The set of elements is 1,2,...,12. The operation is clockwise addition of hours. Each defining property of an abstract group is satisfied, as we shall now demonstrate. First, the "sum" of any two elements is another

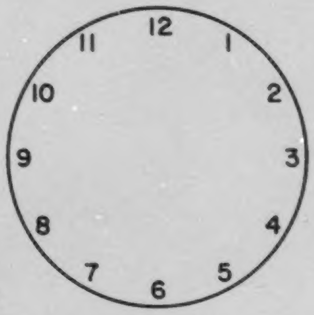


Figure 2-3. A clock face. The addition associated with it gives us a group.

element. For example, we have

$$1 + 6 = 7,$$
$$8 + 4 = 12,$$
$$11 + 2 = 1,$$
$$3 + 12 = 3.$$

Secondly since, for example

$$(8 + 2) + 3 = 1 \quad \text{and} \quad 8 + (2 + 3) = 1,$$

we see that the associative property holds. Thirdly, a full clock rotation, an advance of 12 hours, gives the same time, so that 12 is our unique identity element; thus,

$$12 + 2 = 2 = 2 + 12.$$

Finally, to each of the elements, 1,2,...,12, there corresponds a number we can "add" to obtain 12. Thus

$$4 + 8 = 12 = 8 + 4,$$
$$10 + 2 = 12 = 2 + 10,$$
$$12 + 12 = 12 = 12 + 12.$$

One of the most elegant examples of a group consists of the three cube roots of 1, namely

$$1, \quad \frac{-1 + \sqrt{3}}{2}, \quad \frac{-1 - \sqrt{3}}{2},$$

under multiplication. The demonstration is left to the student as an exercise.

Interestingly enough, although the integers are the most commonly used system that has a group structure (under the operation of addition), they were not the first to have their group structure analyzed. The first groups to be studied extensively were finite groups such as the two examples given above. These groups were found during a study of the theory of equations by Evariste Galois (1811–1832), to whom is credited the origin of the systematic study of Group Theory. Unfortunately, Galois was killed in a duel at the age of 21, immediately after recording some of his most notable theorems.

Exercises 2–6

1. Determine whether the following sets are groups under multiplication:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$;

(b) I, $-I$, K, $-K$,

where

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

2. Show that the set of all elements of M of the form

$$\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}, \text{ where } t \in R \text{ and } t \neq 0,$$

constitutes a group under multiplication.

3. Show that the set of all elements of M of the form

$$\begin{bmatrix} t & s \\ s & t \end{bmatrix}, \text{ where } t \in R, \; s \in R, \text{ and } t^2 - s^2 = 1,$$

constitutes a group under multiplication.

4. If

$$A = \begin{bmatrix} -\frac{1}{2}, & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2}, & -\frac{1}{2} \end{bmatrix},$$

show that the set

$$\{A, A^2, A^3\}$$

is a group under multiplication. Plot the corresponding points in the plane.

5. Let

$$T = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Show that the set

$$\{TIT^{-1}, T(-I)T^{-1}, TKT^{-1}, T(-K)T^{-1}\}$$

is a group under multiplication. Is this true if T is any invertible matrix?

6. Show that the set of all elements of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \text{with} \quad a \in R,\ b \in R, \quad \text{and} \quad ab = 1,$$

is a group under multiplication. If you plot all of the points (a,b), with a and b as above, what sort of a curve do you get?

7. Let

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and let H be the set of all matrices of the form

$$xI + yK, \quad \text{with} \quad x \in R \quad \text{and} \quad y \in R.$$

Prove the following:

(a) The product of two elements of H is also an element of H.

(b) The element $xI + yK$ is invertible if and only if

$$x^2 - y^2 \neq 0.$$

(c) The set of all elements $xI + yK$ with $x^2 - y^2 = 1$ is a group.

8. If a set G of 2×2 matrices is a group, show that each of the following sets are groups:

   (a) $\{A^t \mid A \in G\}$, where A^t = transpose of A;

   (b) $\{B^{-1} AB \mid A \in G\}$, where B is a fixed invertible element of M.

9. If a set G of 2×2 matrices is a group, show that

   (a) $G = \{A^{-1} \mid A \in G\}$,

   (b) $G = \{BA \mid A \in G\}$, where B is any fixed element of G.

10. Using the definition of an abstract group, demonstrate whether or not each of the following sets under the indicated operation is a group:

    (a) the set of odd integers under addition;

    (b) the set of positive real numbers under multiplication;

    (c) the set of the four fourth roots of 1, $\{1, -1, i, -i\}$, under multiplication;

    (d) the set of all integers of the form $3m$, where m is an integer, under addition.

11. By proper application of the four defining postulates of an abstract group, prove that if a, b, and c are elements in a group and $a \circ b = a \circ c$, then $b = c$.

2-7. An Isomorphism between Complex Numbers and Matrices

It is true that very many different kinds of algebraic systems can be expressed in terms of special collections of matrices. Many theorems of this nature have been proved in modern higher algebra. Without attempting any such proof, we shall aim in the present section to demonstrate how the system of complex numbers can be expressed in terms of matrices.

In the preceding section, several subsets of the set of all 2 × 2 matrices were displayed. In particular, the set Z of all matrices of the form

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix}, \quad x \in R \text{ and } y \in R,$$

was considered. We shall exhibit a one-to-one correspondence between the set of all complex numbers, which we denote by C, and the set Z. This one-to-one correspondence would not be particularly significant if it did not preserve algebraic properties — that is, if the sum of two complex numbers did not correspond to the sum of the corresponding two matrices and the product of two complex numbers did not correspond to the product of the corresponding two matrices. There are other algebraic properties that are preserved in this sense.

Usually a complex number is expressed in the form

$$x + yi,$$

where $i = \sqrt{-1}$, $x \in R$, and $y \in R$. Thus, if c is an element of C, the set of all complex numbers, we may write

$$c = x(1) + y(i).$$

The numeral 1 is introduced in order to make the correspondence more apparent. In order to exhibit an element of Z in similar form, we must introduce the special matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I;$$

thus

$$J^2 = -I.$$

The matrix J corresponds to the number i, which satisfies the analogous equation

$$i^2 = -1.$$

This enables us to verify that

$$\begin{aligned} xI + yJ &= x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} \\ &= \begin{bmatrix} x & y \\ -y & x \end{bmatrix}, \end{aligned}$$

which indicates that any element of Z may be written in the form

$$xI + yJ.$$

For example, we have

$$\begin{aligned} 2I + 3J &= 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}, \end{aligned}$$

and

$$0I + 5J = 0\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 5\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 5\\ -5 & 0\end{bmatrix}$$

$$= \begin{bmatrix}0 & 5\\ -5 & 0\end{bmatrix}.$$

Now we can establish a correspondence between C, the set of complex numbers, and Z the set of matrices:

$$x1 + yi \longleftrightarrow xI + yJ.$$

Since each element of C is matched with one element of Z, and each element of Z is matched with one element of C, we call the correspondence one-to-one. Several special correspondences are notable:

$$0 = 0\cdot 1 + 0\cdot i \longleftrightarrow 0\cdot I + 0\cdot J = \underline{0}$$

$$1 = 1\cdot 1 + 0\cdot i \longleftrightarrow 1\cdot I + 0\cdot J = I$$

$$i = 0\cdot 1 + 1\cdot i \longleftrightarrow 0\cdot I + 1\cdot J = J$$

As stated earlier, it is interesting that there is a correspondence between the complex numbers and 2 × 2 matrices, but the correspondence is not particularly significant unless the one-to-one matching is preserved in the operations, especially in addition and multiplication. We shall now follow the correspondence in these operations and demonstrate that the one-to-one property is preserved under the operations.

When two complex numbers are added, the real components are added, and the imaginary components are added. Also, remember that the multiplication of a matrix by a number is distributive; thus, for $a \in R$, $b \in R$, and $A \in M$, we have

$$(a+b)A = aA + bA.$$

Hence we are able to show our one-to-one correspondence under addition:

$$c_1 + c_2 \qquad\qquad Z_1 + Z_2 \quad =$$

$$= (x_1 + iy_1) + (x_2 + iy_2) \qquad (x_1 I + y_1 J) + (x_2 I + y_2 J) =$$

$$= (x_1 + x_2) + (y_1 + y_2)i \longleftrightarrow (x_1 + x_2)I + (y_1 + y_2)J.$$

For example, we have

$$(2 - 3i) + (4 + 1i) \qquad\qquad (2I - 3J) + (4I + 1j) \quad =$$

$$= 6 - 2i \qquad \longleftrightarrow \qquad 6I - 2J.$$

and

$$(3 - 2i) + (2 + 0i) \qquad\qquad (3I - 2J) + (2I + 0J) \quad =$$

$$= 5 - 2i \qquad \longleftrightarrow \qquad 5I - 2J.$$

Before demonstrating that the correspondence is preserved under multiplication, let us review for a moment. An example will suffice:

$$\begin{aligned}(2 + 4i)(3 - 2i) &= 6 - 4i + 12i - 8i^2\\ &= 6 - 8(-1) + (-4 + 12)i\\ &= 14 + 8i;\end{aligned}$$

$$\begin{aligned}(2I + 4J)(3I - 2J) &= 6I^2 - 4IJ + 12JI - 8J^2\\ &= 6I - 4J + 12J - 8(-I)\\ &= 6I + 8I + (-4 + 12)J\\ &= 14I + 8J.\end{aligned}$$

Generally, for multiplication, we have

$$c_1 c_1 \qquad\qquad Z_1 Z_2 \quad =$$

$$= (x_1 + y_1 i)(x_2 + y_2 i) \qquad (x_1 I + y_1 J)(x_2 I + y_2 J) \quad =$$

$$= x_1x_2 + y_1y_2i^2 + x_1y_2i + y_1x_2i \qquad x_1x_2I^2 + y_1y_2J^2 + x_1y_2IJ + y_1x_2JI =$$

$$= (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i \longleftrightarrow (x_1x_2 - y_1y_2)I + (x_1y_2 + x_2y_1)J.$$

If we represent a complex number

$$a + bi$$

as a matrix,

$$a + bi \longleftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

we do have a significant correspondence! Not only is there a one-to-one correspondence between the elements of the two sets, but also the correspondence is one-to-one under the operations of addition and multiplication.

The additive and multiplicative identity elements are, respectively,

$$0 = 0 + 0i \longleftrightarrow \begin{bmatrix} 0 & 0 \\ -0 & 0 \end{bmatrix} = \underline{0}$$

and

$$1 = 1 + 0i \longleftrightarrow \begin{bmatrix} 1 & 0 \\ -0 & 1 \end{bmatrix} = I;$$

and for the additive inverse of

$$a + bi \longleftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

we have

$$- a - bi \longleftrightarrow \begin{bmatrix} -a & -b \\ b & -a \end{bmatrix}.$$

Let us now examine how the multiplicative inverses, or reciprocals, can be matched. We have seen that any member of the set of 2×2 matrices has a multiplicative inverse if and only if the determinant does not equal zero. That is, if $A \in Z$ then there exists A^{-1} if and only if $x^2 + y^2 \neq 0$ since

$\delta(A) = x^2 + y^2$ if $A = xI + yJ$. Now we know that any complex number has a multiplicative inverse, or reciprocal, if and only if the complex number is not zero. That is, if $c = x + yi$, then there exists a multiplicative inverse if and only if $x + yi \neq 0$, which means that x and y are not both 0. This is equivalent to saying that $x^2 + y^2 \neq 0$, since $x \in R$ and $y \in R$. For multiplicative inverses, if

$$x^2 + y^2 \neq 0,$$

our correspondence yields

$$c_1 = x + yi \qquad \longleftrightarrow \qquad xI + yJ = Z_1$$

$$\frac{1}{c_1} = \frac{1}{x^2 + y^2}(x - yi) \longleftrightarrow \frac{1}{x^2 + y^2}(xI - yJ) = Z_1^{-1}$$

It is now clear that the correspondence between C, the set of complex numbers, and Z, a subset of all 2 × 2 matrices,

$$x + yi \longleftrightarrow xI + yJ,$$

is preserved under the algebraic operations. All of this may be summed up by saying that C and Z have identical algebraic structures. Another way of expressing this is to say that C and Z are isomorphic. This word is derived from two Greek words and means "of the same form." Two number systems are isomorphic if, first, there is a mapping of one onto the other that is a one-to-one correspondence and, secondly, the mapping preserves sums and products. If two number systems are isomorphic, their structures are the same; it is only their terminology that is different. The world is heavy with examples of isomorphisms, some of them trivial and some quite the opposite. One of the simplest is the isomorphism between the natural numbers and the positive integers, a subset of the integers; another is that between the real numbers and the subset $a + 0i$ of all complex numbers. (We should quickly guess that there is an isomorphism between real numbers a and the set of all matrices of the form $aI + 0J$!)

An example of an isomorphism that is more difficult to understand is that between real numbers and residue classes of polynomials. We won't try to explain this one; but there is one more fundamental concept that can be in introduced here, as follows.

We have stated that the real numbers are isomorphic to a subset of the complex numbers. We speak of the algebra of the real numbers as being embedded in the algebra of complex numbers. In this sense, we can say that the algebra of complex numbers is embedded in the algebra of 2×2 matrices. Also, we can speak of the complex numbers as being "richer" than the real numbers, or of the 2×2 matrices as being richer than the complex numbers. The existence of complex numbers gives us solutions to equations such as

$$x^2 + 1 = 0,$$

which have no solution in the domain of real numbers. It is of course clear that Z is a proper subset of M, that is, $Z \subset M$ and $Z \neq M$. Here is a simple example to illustrate the statement that M is "richer" than Z: The equation

$$X^2 - I = 0$$

has for solution any matrix

$$X = \begin{bmatrix} 0 & t \\ 1/t & 0 \end{bmatrix}, \quad t \in R \text{ and } t \neq 0,$$

as may be seen quickly by easy computation. On the other hand, the equation

$$x^2 - 1 = 0$$

has exactly two solutions among the complex numbers, namely $c = 1$ and $c = -1$.

Exercises 2-7

1. Using the following values, show the correspondence under addition and multiplication between complex numbers of the form $x + yi$ and matrices of the form $xI + yJ$:

   (a) $x_1 = 1$, $y_1 = -1$, $x_2 = 0$, and $y_2 = -2$;

   (b) $x_1 = 3$, $y_1 = -4$, $x_2 = 1$, and $y_2 = 1$;

   (c) $x_1 = 0$, $y_1 = -5$, $x_2 = 3$, and $y_2 = 4$.

2. Carry through, in parallel columns as in the text, the necessary computations to establish an isomorphism between R and the set

$$N = \left\{ \begin{bmatrix} x, & 0 \\ 0, & x \end{bmatrix} : x \in R \right\}$$

by means of the correspondence

$$x \longleftrightarrow \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}.$$

3. In the preceding exercise, an isomorphism between R and the sets of matrices

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, \quad x \in R,$$

was considered. Define a function

$$f : R \longrightarrow M$$

by

$$f(x) = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}.$$

Determine which of the following statements are correct:

(a) $f(x + y) = f(x) + f(y)$,

(b) $f(xy) = \left[f(x)\right] \left[f(y)\right]$,

(c) $f(0) = \underline{0}$,

(d) $f(1) = I$

(e) $f\left(\frac{1}{x}\right) = (f(x))^{-1}$, $x \neq 0$.

4. Is the set of matrices

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

with a and b rational and with $a^2 + b^2 = 1$, a group under multiplication?

2-8. Algebras

The concepts of group, ring, and field are of frequent occurrence in modern algebra. The study of these systems is a study of the structures or patterns that are the framework on which algebraic operations are dependent. In this chapter, we have attempted to demonstrate how these same concepts describe the structure of the set of 2×2 matrices, which is a subset of the set of all rectangular matrices.

Not only have we introduced these embracing concepts, but we have exhibited the "algebra" of the sets. "Algebra" is a generic word that is frequently used in a loose sense. By technical definition, an algebra is a system that has two binary operations, called "addition" and "multiplication," and also has "multiplication by a number," that make it both a ring and a vector space.

Vector spaces will be discussed in Chapter 4 and we shall see then that the set of 2×2 matrices constitutes a vector space under matrix addition and multiplication by a number. Thus the 2×2 matrices form an algebra.

As you yourself might conclude at this time, this algebra is only one of many possible algebras. Some of these algebras are duplicates of one another in the sense that the basic structure of one is the same as the basic structure of another. Superficially, they seem different because of the terminology. When they have the same structure, two algebras are called isomorphic. One of the interesting observations about modern mathematics is that the structure of these new branches often overlaps parts of the old mathematics with which we are already familiar.

Chapter 3

MATRICES AND SYSTEMS OF LINEAR EQUATIONS

3-1. Introduction

In this chapter, we are going to demonstrate the use of matrices in the solution of systems of linear equations. We shall first review a few well-known algebraic techniques for solving these systems and then shall show how some of the same techniques can be duplicated in terms of the matrix operations with which you are now familiar.

Our study will thus lead us naturally into the application of matrices to the solution of systems of linear equations.

In your previous study of algebra, you probably learned several methods for seeking solutions of such systems of linear equations as

$$\begin{aligned} 2x - \ \ y &= 3, \\ -5x + 3y &= -7. \end{aligned} \tag{1}$$

Thus, you might recall the method of substitution and the method of elimination.

For example, you can solve the first of the above equations (1) for y in terms of x,

$$y = 2x - 3,$$

substitute this expression in the second equation, obtaining

$$-5x + 3(2x - 3) = -7,$$

whence

$$x = 2 \tag{2}$$

and accordingly

122

$$y = 2x - 3 = 2(2) - 3 = 1.$$

Or you can multiply both members of the first equation in (1) by 3 to obtain

$$6x - 3y = 9,$$

add this to the second equation in (1) to eliminate y, getting

$$x = 2, \qquad (2)$$

whence

$$y = 1$$

as before.

In each of the foregoing procedures, what has actually been demonstrated is only that if there is a solution set of values (x,y) for the system (1), then $(x,y) = (2,1)$. For logical completeness, you should substitute these values in the original equations (1) and observe that for them the equations are valid statements:

$$2(2) - 1 = 3, \quad 5(2) + 3(1) = -7.$$

Alternatively, of course, for logical completeness you might demonstrate that each of the steps you have taken is "reversible"—that is, that the validity of each new system of equations implies that of the former system—so that finally the system of equations,

$$x = 2, \quad y = 1,$$

with which you ended is equivalent to the original system; that is, every solution of one system is a solution of the other, and conversely.

For example, the system

$$\begin{aligned} 2x - y &= 3, \\ x &= 2, \end{aligned} \qquad (3)$$

which consists of Equation (2) and the first equation in (1), was obtained by means of algebraic operations from the system (1). Accordingly, any solution of the system (1) is also a solution of the system (3). Conversely, the validity of the system (3) implies that of the original system (1), since the first equation in (1) is included also in (3), and since the second equation in (1) results from subtracting 3 times the first equation in (3) from the second equation in (3). Accordingly, the two systems are equivalent.

Direct verification by the substitution of $x = 2$ and $y = 1$ in the original equations (1) has the advantage, however, that it guards against computational errors.

In the present chapter, we shall investigate two routine and orderly methods of elimination, without regard to the particular values of the coefficients except that we shall avoid division by 0. The first of these, the triangulation method, is an extremely efficient general way of solving a single system of equations. The diagonal method, which is treated next, is an extension of the triangulation method. It is rather less efficient than the triangulation method in solving a single system; but it is especially useful in dealing with several systems in which corresponding coefficients of the variables are equal while the right-hand members are different—a situation that often occurs in industrial and applied scientific problems.

The triangulation method and the diagonal method are procedures of the sort you might use, for example, in "programming," i.e., devising a method, or program, for solving a system of linear equations by means of a modern electronic computing machine. Before long, these "magic brains" may be developed to the point where they are able even to choose for themselves the most efficient method for dealing with any particular set of coefficients.

The methods will lead you naturally to see how the theory of matrices that you have been studying is directly applicable to the solution of these systems. In particular, you will see how the diagonal method can be used in matrix inversion and how very useful the inverse of a matrix is in the solution of systems of linear equations.

Exercises 3-1

1. Solve the following systems of equations:

(a)
$$\begin{aligned} 3x + 4y &= 4, \\ 5x + 7y &= 1; \end{aligned}$$

(b)
$$\begin{aligned} x - 2y &= 3, \\ y &= 2; \end{aligned}$$

(c)
$$\begin{aligned} x + y - z &= 3, \\ 2y + z &= 10, \\ 5x - y - 2z &= -3; \end{aligned}$$

(d)
$$\begin{aligned} x - 3y + 2z &= 6, \\ y - z &= -4, \\ z &= 7; \end{aligned}$$

(e)
$$\begin{aligned} x + 2y + z - 3w &= 2, \\ y - 2z + w &= 7, \\ z - 2w &= 0, \\ w &= 3; \end{aligned}$$

(f)
$$\begin{aligned} 1x + 0y + 0z + 0w &= a, \\ 0x + 1y + 0z + 0w &= b, \\ 0x + 0y + 1z + 0w &= c, \\ 0x + 0y + 0z + 1w &= d. \end{aligned}$$

2. Solve by drawing graphs:

(a)
$$\begin{aligned} x + y &= 2, \\ x - y &= 2; \end{aligned}$$

(b)
$$\begin{aligned} 3x + 4y &= 1, \\ 5x + 7y &= 1. \end{aligned}$$

3. Which of the following statements is correct? Which of the final conclusions is actually valid? If

$$4^2 - 2 \cdot 4 \cdot 5 + 5^2 = 5^2 - 2 \cdot 5 \cdot 4 + 4^2,$$

then

$$(4 - 5)^2 = (5 - 4)^2,$$

so that

$$4 - 5 = 5 - 4,$$

whence

$$-1 = 1.$$

If

$$-1 = 1,$$

then

$$4 - 5 = 5 - 4,$$

so that

$$(4 - 5)^2 = (5 - 4)^2,$$

whence

$$4^2 - 2\cdot4\cdot5 + 5^2 = 5^2 - 2\cdot5\cdot4 + 4^2.$$

3-2 The Triangulation Method

The triangulation method for solving systems of linear equations is best presented by example. The method consists of a step-by-step replacement of a given system by a sequence of simpler but equivalent systems.

Consider, for example, the system

$$\begin{aligned} 3x + 2y - 2z &= 3, \\ 2x - y - 4z &= 4, \\ -x + y + 5z &= 0. \end{aligned} \qquad \text{(I)}$$

The basic objective of the triangulation method is to replace such a system as (I) by an equivalent system of the form

$$\begin{aligned} x + b_1 y + c_1 z &= d_1, \\ y + c_2 z &= d_2, \\ z &= d_3, \end{aligned}$$

if this is possible. The value for z is then substituted from the third equation into the second to determine a value for y, and then both of these values are substituted into the first equation to determine a value for x.

For a system such as (I), the procedure in achieving the basic objective—if it can be carried out—is first to obtain the desired coefficients 1, 0, 0 for x, next the desired coefficients 1, 0 for y, and then the coefficient 1 for z. Thus this procedure might be said to consist of 3 "molecules," of 3, 2, and 1 "atoms," respectively.

For the solution of the system (I), the first molecule has 3 atoms: (i) the first equation in (I) is multiplied by 1/3 to yield

$$x + \frac{2}{3}y - \frac{2}{3}z = 1; \tag{1}$$

(ii) −2/3 times the first equation is added to the second equation in (I) to get

$$-\frac{7}{3}y - \frac{8}{3}z = 2; \tag{2}$$

and (iii) 1/3 times the first equation is added to the third equation in (I) to obtain

$$\frac{5}{3}y + \frac{13}{3}z = 1. \tag{3}$$

Now Equations (1), (2), and (3) constitute the system

$$\begin{aligned} x + \frac{2}{3}y - \frac{2}{3}z &= 1, \\ -\frac{7}{3}y - \frac{8}{3}z &= 2, \\ \frac{5}{3}y + \frac{13}{3}z &= 1. \end{aligned} \tag{II}$$

Thus, any solution of the system (I) is also a solution of the system (II).

On the other hand, by reversing the atomic process, you can show for yourself that the reverse implication also holds; that is, any solution of the system (II) is also a solution of the system (I). Accordingly, the two systems are equivalent: Every solution of one of the systems is a solution of the other, and vice versa.

The second molecule has two atoms. Namely, (i) the second equation in (II) is multiplied by $-3/7$ to obtain

$$y + \frac{8}{7} z = -\frac{6}{7} ;$$

and (ii) 5/7 times the second equation is added to the third equation in (II) to yield

$$\frac{17}{7} z = \frac{17}{7} .$$

We now have the equivalent system

$$\begin{aligned} x + \frac{2}{3} y - \frac{2}{3} z &= 1 , \\ y + \frac{8}{7} z &= -\frac{6}{7} , \\ \frac{17}{7} z &= \frac{17}{7}. \end{aligned} \tag{III}$$

The third molecule has just one atom: we multiply the third equation of the present system by 7/17 and thus obtain the equivalent system

$$\begin{aligned} x + \frac{2}{3} y - \frac{2}{3} z &= 1, \\ y + \frac{8}{7} z &= -\frac{6}{7} , \\ z &= 1 . \end{aligned} \tag{IV}$$

We have now completed what is known as the *forward solution* of the system (I). For the *backward solution*, we substitute $z = 1$ into the second equation in (IV), getting

$$y = -2,$$

and then substitute $z = 1$ and $y = -2$ in the first equation, finally obtaining

$$x = 3.$$

In the backward solution, the systems we have obtained are still equivalent to the original system. Thus, if we have made no computational mistakes, we have determined that the system (I) has the unique solution

$$(x, y, z) = (3, -2, 1).$$

To make the steps of the triangulation method quite clear, let us detach the coefficients of x, y, and z in the system (I), thus:

$$\begin{array}{rrrcr} x & y & z & & \\ \hline 3 & 2 & -2 & = & 3 \\ 2 & -1 & -4 & = & 4 \\ -1 & 1 & 5 & = & 0 \end{array} .$$

The symbols x, y, z have been placed in a row at the top of the columns to serve as a memory device; in the next section, when we shall be working with matrices, they will appear as a column on the right.

In the foregoing process, what we sought was an equivalent system containing coefficients 0 and 1 as follows:

$$\begin{array}{|ccc|} \multicolumn{1}{c}{x} & y & \multicolumn{1}{c}{z} \\ \hline 1 & b_1 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \\ \hline \end{array} \begin{array}{c} \\ = \\ = \\ = \end{array} \begin{array}{|c|} \multicolumn{1}{c}{} \\ \hline d_1 \\ d_2 \\ d_3 \\ \hline \end{array} .$$

What would we have sought if the system had consisted of 2 equations in 2 variables? of 4 equations in 4 variables? Can you suggest why this might be called the "triangulation" method?

In terms of detached coefficients, the steps in the foregoing forward triangulation solution of the system (I) went like this:

$$\begin{array}{|ccc|} \multicolumn{1}{c}{x} & y & \multicolumn{1}{c}{z} \\ \hline 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \\ \hline \end{array} \begin{array}{c} \\ = \\ = \\ = \end{array} \begin{array}{|c|} \multicolumn{1}{c}{} \\ \hline 3 \\ 4 \\ 0 \\ \hline \end{array} \Rightarrow \begin{array}{|ccc|} \multicolumn{1}{c}{x} & y & \multicolumn{1}{c}{z} \\ \hline 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & -\frac{7}{3} & -\frac{8}{3} \\ 0 & \frac{5}{3} & \frac{13}{3} \\ \hline \end{array} \begin{array}{c} \\ = \\ = \\ = \end{array} \begin{array}{|c|} \multicolumn{1}{c}{} \\ \hline 1 \\ 2 \\ 1 \\ \hline \end{array}$$

$$\Rightarrow \begin{array}{|ccc|} \multicolumn{1}{c}{x} & y & \multicolumn{1}{c}{z} \\ \hline 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & \frac{17}{7} \\ \hline \end{array} \begin{array}{c} \\ = \\ = \\ = \end{array} \begin{array}{|c|} \multicolumn{1}{c}{} \\ \hline 1 \\ -\frac{6}{7} \\ \frac{17}{7} \\ \hline \end{array} \tag{4}$$

$$\Rightarrow \begin{array}{|ccc|} \multicolumn{1}{c}{x} & y & \multicolumn{1}{c}{z} \\ \hline 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & 1 \\ \hline \end{array} \begin{array}{c} \\ = \\ = \\ = \end{array} \begin{array}{|c|} \multicolumn{1}{c}{} \\ \hline 1 \\ -\frac{6}{7} \\ 1 \\ \hline \end{array}$$

Generally, for the system

$$\begin{aligned} a_{11}\,x + a_{12}\,y + a_{13}\,z &= d_1, \\ a_{21}\,x + a_{22}\,y + a_{23}\,z &= d_2, \\ a_{31}\,x + a_{32}\,y + a_{33}\,z &= d_3, \end{aligned}$$

the triangulation method proceeds like this if none of the coefficients we want to divide by are 0:

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \boxed{\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}} \end{array} \begin{array}{c} \\ = \\ = \\ = \end{array} \begin{array}{c} \\ \boxed{\begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array}} \end{array} \Rightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \boxed{\begin{array}{ccc} 1 & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{array}} \end{array} \begin{array}{c} \\ = \\ = \\ = \end{array} \begin{array}{c} \\ \boxed{\begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array}} \end{array}$$

$$\Rightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \boxed{\begin{array}{ccc} 1 & b_{12} & b_{13} \\ 0 & 1 & c_{23} \\ 0 & 0 & c_{33} \end{array}} \end{array} \begin{array}{c} \\ = \\ = \\ = \end{array} \begin{array}{c} \\ \boxed{\begin{array}{c} f_1 \\ g_2 \\ g_3 \end{array}} \end{array} \qquad (5)$$

$$\Rightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \boxed{\begin{array}{ccc} 1 & b_{12} & b_{13} \\ 0 & 1 & c_{23} \\ 0 & 0 & 1 \end{array}} \end{array} \begin{array}{c} \\ = \\ = \\ = \end{array} \begin{array}{c} \\ \boxed{\begin{array}{c} f_1 \\ g_2 \\ h_3 \end{array}} \end{array}$$

Can you express each of the b's in terms of the a's? each of the c's in terms of the b's? What are similar expressions for the f's, g's, and h_3?

Exercises 3–2

1. In general, in the triangulation method for solving a system of four linear equations in four variables, how many "molecules" are there in the forward solution? How many "atoms" in each of the "molecules?" How many individual additions and multiplications in the forward and backward solutions together?

2. Solve the following systems of linear equations by the triangulation method:

(a) $3x = 4;$

(b)
$$\begin{aligned} x - y &= 3, \\ x + y &= 4; \end{aligned}$$

(c)
$$\begin{aligned} 2x - y + z &= -1, \\ 3x + 2y + 3z &= 3, \\ x + y + z &= 2; \end{aligned}$$

(d)
$$\begin{aligned} x + y + z + w &= 9, \\ x - y - z + w &= -1, \\ x - y + z - w &= -3, \end{aligned}$$

3. The solution set of one of the following systems of linear equations is empty, while the other solution set contains an infinite number of solutions. See if you can determine by the triangulation method which is which, and give three particular numerical solutions for the system that does have solutions:

(a)
$$\begin{aligned} x + 2y - z &= 3, \\ x - y + z &= 4, \\ 4x - y + 2z &= 14; \end{aligned}$$

(b)
$$\begin{aligned} x + 2y - z &= 3, \\ x - y + z &= 4, \\ 4x - y + 2z &= 15. \end{aligned}$$

4. For the scheme (5) of this section, express the b's in terms of the a's, the c's in terms of the b's, the f's in terms of the a's and e's, the g's in terms of the f's and b's, and h_3 in terms of the a's and e's.

3–3. Formulation in Terms of Matrices

In this section and the next, we shall see how the matrix notation and operations that were developed in Chapter 1 can be used to write a system of linear equations in matrix form and to carry out the steps of the triangulation method for solving the system.

First, for the system (I) of Section 3-2, namely,

$$\begin{aligned} 3x + 2y - 2z &= 3, \\ 2x - y - 4z &= 4, \\ -x + y + 5z &= 0, \end{aligned} \tag{I}$$

let us consider the array of detached coefficients of x, y, and z as a matrix,

$$A = \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix}.$$

Next, let us consider the column matrices, or column vectors,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix},$$

whose entries also occur in that system of equations. By the definition of matrix multiplication, we have

$$AX = \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x + 2y - 2z \\ 2x - y - 4z \\ -x + y + 5x \end{bmatrix},$$

which is a column matrix whose entries are the left-hand members of the equations of our linear system (I).

Now the equation

$$AX = B, \tag{1}$$

that is,

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix},$$

is equivalent, by the definition of the equality of matrices, to the entire system of linear equations (I). It is an achievement not to be taken modestly that we are able to consider, and work with, a large system of equations in terms of such a simple representation as is exhibited in Equation (1). A pattern is beginning to emerge, but we shall not now spoil the fun by announcing the final results.

There is an interesting way of viewing the product

$$AX = \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x + 2y - 2z \\ 2x - y - 4z \\ -x + y + 5z \end{bmatrix} = Y.$$

You will recall that the equations you have been handling earlier, such as

$$y = ax + b$$

and

$$y = \sin x,$$

express functions, or mappings, with numbers x constituting the domain and numbers y constituting the range. The above matrix A can also be considered as determining a function, with the variable X on a domain of column matrices, and with the variable

$$Y = \begin{bmatrix} 3x + 2y - 2z \\ 2x - y - 4z \\ -x + y + 5z \end{bmatrix}$$

also on a range of column matrices: a matrix function of a matrix variable! We have not previously considered functions of this sort.

In terms of matrix functions, what is the meaning of Equation (1)? The matrix function determined by the matrix A maps the X domain of matrices onto a Y range of matrices. Equation (1) asks an inverse question: What matrix or matrices X (if any) are mapped on the particular matrix B? Of course, we have already found in Section 3-2 that the unique valid answer to this question is

$$X = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} .$$

We shall consider some geometric aspects of this matrix-function point of view in Chapters 4 and 5.

Now look again at the scheme (4) in Section 3-2, but this time in terms of matrices:

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & -\frac{7}{3} & -\frac{8}{3} \\ 0 & \frac{5}{3} & \frac{13}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & \frac{17}{7} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{6}{7} \\ \frac{17}{7} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{6}{7} \\ 1 \end{bmatrix}$$

We already know from the work of Section 3-2 that these matrix equations are equivalent, so that the implication arrows $\Rightarrow$ can be replaced by two-headed arrows $\Leftrightarrow$.

Our present concern, however, is the question: Can the foregoing implications be achieved through matrix operations? This question will be treated in the next section.

Exercises 3-3

1. Perform the indicated multiplications:

$$\text{(a)}\quad \begin{bmatrix} 4 & -2 & 7 \\ 3 & 1 & 5 \\ 0 & 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \text{(b)}\quad \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix}.$$

2. Write in matrix form:

$$\text{(a)}\quad \begin{aligned} 4x - 2y + 7z &= 2, \\ 3x + y \neq 5z &= -1, \\ 6y - z &= 3; \end{aligned} \qquad \text{(b)}\quad \begin{aligned} x + y &= 2, \\ x - y &= 2. \end{aligned}$$

3. Determine the systems of algebraic equations to which the matrix equations,

$$\text{(a)}\quad \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \text{(b)}\quad \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{bmatrix},$$

are equivalent.

4. Onto what vector does the function defined by

$$Y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

map the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$? What vector does it map onto the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$?

5. Perform the following matrix multiplications:

$$\text{(a)}\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

$$\text{(b)}\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$,

(d) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

3-4. Solution by Means of Matrices

In applying the triangulation method to the solution of the system (I) of Section 3-2, namely, to

$$\begin{aligned} 3x + 2y - 2z &= 3, \\ 2x - y - 4z &= 4, \\ -x + y + 5z &= 0, \end{aligned} \tag{I}$$

we carried out just two types of algebraic operations in obtaining an equivalent system in triangular form:

(a) multiply an equation by a number other than 0;

(b) add an equation to another equation.

A third type of operation is sometimes required, namely:

(c) interchange two equations.

This third operation would have been necessary if a coefficient by which we otherwise would have divided had happened to be 0, and there had been a subsequent equation in which the same variable had a nonzero coefficient.

The three foregoing operations can, in effect, be carried out through matrix multiplication. We shall illustrate this statement through examples involving a given 3×3 matrix of coefficients of the variables x, y, and z, namely,

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

You can easily see, however, that the comments that follow hold more generally for an arbitrary rectangular matrix of coefficients.

(a') Consider the product

$$\begin{bmatrix} n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} na & nb & nc \\ d & e & f \\ g & h & i \end{bmatrix}.$$

You should perform this multiplication yourself to see that the result is correct. The operation has the effect of multiplying the first row of A by n. To multiply the second or third row by n, you can verify that you would multiply on the left by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n \end{bmatrix},$$

respectively. Thus, to multiply the p—th row of a matrix A by n, you multiply A on the left by a matrix J obtained from the identity matrix I through multiplying the p—th row of I by n.

(b') Consider the product

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d+g & e+h & f+i \\ g & h & i \end{bmatrix};$$

this multiplication has the effect of adding the third row of A to the second row of A. To add the third row to the first, for example, or the first to the second, you would multiply on the left by

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

respectively. Thus, to add the p—th row of A to the q—th row of A, you multiply A on the left by a matrix K obtained from the identity matrix I by adding the p-th row of I to the q-th row of I.

(c') Consider the products

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}.$$

Thus you see that to interchange the p-th and q-th rows of a matrix A, you multiply A on the left by a matrix L obtained from the identity matrix I by interchanging the p-th and q-th rows of I.

Definition 3-1. The matrix multipliers J, K, and L described in paragraphs (a'), (b'), and (c'), above, are called the elementary matrices.

The foregoing rules for determining elementary matrices J, K, and L, to be used in operating on the rows of a matrix A through matrix multiplication, are extremely easy to remember: You simply perform the desired operation on I instead of A. You might note, however, that these operations on I are not operations defined in our algebra of matrices; they are merely devices for constructing the left-hand multiplier J, K, or L.

Each elementary matrix E (that is, each J, K or L) has an inverse, that is, a matrix E^{-1} such that

$$E^{-1} E = I = E E^{-1}.$$

For example, the inverses of the elementary matrices

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the elementary matrices

$$J^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{n} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad L^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

respectively, as you can verify either by performing the multiplications that are involved or by recalling the effect of multiplying any matrix A by one of these elementary matrices. Thus, the above matrix L differs from the identity matrix I in having its first two rows interchanged; and multiplying on the left by L' has the effect of interchanging the first two rows.

Multiplications by elementary matrices can be combined. Thus, to multiply the first row of A by $1/3$, we would multiply A on the left by the elementary matrix J with $a_{11} = 1/3$:

$$J = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

and to add $-2/3$ times the first row to the second row, or $1/3$ times the first row to the third, we would multiply A on the left by the product of elementary matrices of type $J^{-1} K J$:

$$\begin{bmatrix} -\frac{3}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

or

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix},$$

respectively. To perform all three of these operations at the same time, we would multiply A on the left by

$$M_1 = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix},$$

which can similarly be shown to be a product of elementary matrices.

Now the three operations performed on the matrix A through multiplying A on the left by the above matrix M_1 correspond precisely to the three atoms (i), (ii),, and (iii) of the first molecule in the triangulation solution of the system (I); see page 120.

In matrix form, the system (I) is

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}.$$

We multiply both sides of this equation on the left by M_1, thus:

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}. \tag{1}$$

If you will carry out the numerical computations both in the left-hand member and in the right-hand member of Equation (1), you will obtain the anticipated result:

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & -\frac{7}{3} & -\frac{8}{3} \\ 0 & \frac{5}{3} & \frac{13}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} .$$

This is the matrix form of the system (II) of Section 3-2.

If you look now in Section 3-2 at the two atoms (i) and (ii) of the second molecule in the solution of the system (I), you will ascertain that the corresponding matrix multiplier must be

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{7} & 0 \\ 0 & \frac{5}{7} & 1 \end{bmatrix} ,$$

since this time we want to multiply the second row of the matrix of coefficients by −3/7 and to add 5/7 times the second row to the third. The matrix M_2 applies thus:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{7} & 0 \\ 0 & \frac{5}{7} & 2 \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & -\frac{7}{3} & -\frac{8}{3} \\ 0 & \frac{5}{3} & \frac{13}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{7} & 0 \\ 0 & \frac{5}{7} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} ,$$

to yield

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & \frac{17}{7} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{6}{7} \\ \frac{17}{7} \end{bmatrix} ,$$

which is the matrix form of the system (III) of Section 3-2.

The third molecule, with its one atom, has the corresponding matrix multiplier

$$M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{17} \end{bmatrix},$$

since it leaves the first equation unaltered, leaves the second equation unaltered, and multiplies the third equation by 7/17; applying M_3, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{17} \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & \frac{17}{7} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{17} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{6}{7} \\ \frac{17}{7} \end{bmatrix},$$

or

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{6}{7} \\ 1 \end{bmatrix}. \qquad (2)$$

Now (2), of course, is precisely the matrix version of the equivalent triangulated system (IV) of Section 3-2:

$$\begin{aligned} x + \tfrac{2}{3}y - \tfrac{2}{3}z &= 1, \\ y + \tfrac{8}{7}z &= -\tfrac{6}{7}, \\ z &= 1, \end{aligned}$$

from which the backward solution yields $(x, y, z) = (1, -2, 3)$ as before.

To review the foregoing process, and to visualize the operations more generally, we might note that the successive matrix multipliers for the scheme (5) of Section 3-2 are

$$\begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{b_{22}} & 0 \\ 0 & -\frac{b_{32}}{b_{22}} & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{c_{33}} \end{bmatrix}.$$

Let us now take advantage of the associative law for the multiplication of matrices to form the product

$$M = M_3M_2M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{17} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{7} & 0 \\ 0 & \frac{5}{7} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{7}{17} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{7} & -\frac{3}{7} & 0 \\ -\frac{1}{7} & \frac{5}{7} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{7} & -\frac{3}{7} & 0 \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix}.$$

If M is applied to the original system of equations (I) in matrix form, thus;

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{7} & -\frac{3}{7} & 0 \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix} \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{7} & -\frac{3}{7} & 0 \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix},$$

then the equivalent triangulated system (2) is obtained directly, as you can verify by performing the numerical computation.

If we are confronted with a second system of linear equations, say

$$\begin{aligned} 3x + 2y - 2z &= -2, \\ 2x - y - 4z &= -12, \\ -x + y + 5z &= 18, \end{aligned}$$

in which the coefficients of the variables are the same as in the original system (I), while the right-hand numbers are different, then we can again apply the same matrix M to obtain the same triangulated matrix of detached coefficients, thus:

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{7} & -\frac{3}{7} & 0 \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix} \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{7} & -\frac{3}{7} & 0 \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix} \begin{bmatrix} -2 \\ -12 \\ 18 \end{bmatrix},$$

or

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{32}{7} \\ 4 \end{bmatrix},$$

whence the backward-solution process yields

$$(x, y, z) = (2, 0, 4).$$

When we are confronted with the problem of solving two or more systems of linear equations that differ only in their right-hand members, however, it is advantageous to effect a further simplification of the matrix of detached coefficients, obviating the necessity of performing the backward solution each time. This will be done through the diagonal process in the next section.

Exercises 3-4

1. Solve the following systems of equations by the triangulation method:

$$\text{(a)}\quad \begin{aligned} 3x + 2y - 2z &= -4,\\ 2x - y - 4z &= 2,\\ -x + y + 5z &= 7; \end{aligned} \qquad \text{(b)}\quad \begin{aligned} x - y - 2z &= 3,\\ y + 3z &= 5,\\ 2z + 2y - 3z &= 15. \end{aligned}$$

2. Solve by the triangulation method:

$$\text{(a)}\quad \begin{bmatrix} 1 & 4 & 7\\ 2 & 3 & 6\\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 3\\ 0\\ 7 \end{bmatrix},$$

$$\text{(b)}\quad \begin{bmatrix} 1 & 4 & 7\\ 2 & 3 & 6\\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 3\\ 0\\ -2 \end{bmatrix},$$

$$\text{(c)}\quad \begin{bmatrix} 1 & 4 & 7\\ 2 & 3 & 6\\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 1\\ 3\\ -4 \end{bmatrix},$$

$$\text{(d)}\quad \begin{bmatrix} 1 & 4 & 7\\ 2 & 3 & 6\\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}.$$

3. Solve by the triangulation method:

$$\begin{bmatrix} 4 & 0 & 2\\ 1 & 3 & 1\\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} x & u\\ y & v\\ z & w \end{bmatrix} = \begin{bmatrix} 4 & 2\\ 1 & 6\\ 3 & 7 \end{bmatrix}.$$

3-5. The Diagonal Method

For the forward triangulation solution of the system

$$\begin{bmatrix} 3 & 2 & -2\\ 2 & -1 & -4\\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 3\\ 4\\ 0 \end{bmatrix}, \tag{I}$$

you will recall that in Section 3-4 we sought an equivalent matrix equation having coefficient matrix of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \tag{1}$$

without regard to the unspecified entires a, b, and c in the upper right-hand portion. After this was achieved, the backward solution was employed to obtain the answer:

$$(x, y, z) = (3, -2, 1).$$

In case we have to solve several systems of linear equations, all with the same coefficient matrix but with different right-hand members, as in some of the problems at the end of the preceding section, it is more efficient to obtain an equivalent system with the identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

as coefficient matrix, for then the backward-solution procedure is eliminated. It is always possible to obtain a coefficient matrix of the form (2) if it is possible to obtain one of the form (1).

You should recall that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = X,$$

in order to appreciate the value of having a coefficient matrix of the form (2). Can you tell how it is that no backward solution is needed in this case? Can you suggest why a method of solution involving a matrix of the form (2) might be called the "diagonal method"? (The 1's in Equation (2) are on the principal diagonal of the matrix.) What matrix corresponds to (2) if the system consists of 2 equations in 2 variables? of 4 equations in 4 variables?

Of course, more work is required to obtain a coefficient matrix of the form (2) than is required to obtain one of the less special form (1). You will recall that in obtaining an equivalent system with coefficient matrix of the form (1), the procedure consists ordinarily of 3 "molecules," of 3, 2, and 1 "atoms," respectively. In obtaining a coefficient matrix of the form (2), as you will see, the procedure again ordinarily consists of 3 molecules, but now each molecule contains 3 atoms; however, in general the additional work in obtaining a coefficient matrix of the form (2) is more than compensated for even if there are only 2 systems with identical coefficient matrices to be solved.

The diagonal method differs from, and extends, the triangulation method as follows: whereas in the triangulation method we seek to obtain a coefficient matrix with 1's all along the principal diagonal and with 0's everywhere below this diagonal, in the diagonal method we seek to obtain a coefficient matrix with 0's also above the principal diagonal. The way to determine the matrix multipliers in order to do this should be apparent from a review of the rules given in Section 3-4; this will be illustrated in the next section.

Exercises 3-5

1. Perform the following multiplications:

(a) $$\begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{17} & \frac{12}{17} & \frac{10}{17} \\ \frac{6}{17} & -\frac{13}{17} & -\frac{8}{17} \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix},$$

(b) $$\begin{bmatrix} \frac{1}{17} & \frac{12}{17} & \frac{10}{17} \\ \frac{6}{17} & -\frac{13}{17} & -\frac{8}{17} \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix} \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix}.$$

2. Multiply both members of the matrix equation

$$\begin{bmatrix} 1 & 12 & 10 \\ 6 & -13 & -8 \\ -1 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ 0 \\ 0 \end{bmatrix}$$

on the left by

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix},$$

and use the result to solve the equation.

3. Solve the following system of linear equations:

$$\begin{aligned} x + 12y + 10z &= -9, \\ 6x - 13y - 8z &= 31, \\ -x + 5y + 7z &= -8. \end{aligned}$$

3–6. Matrix Inversion

Let us apply the diagonal method to the system (I) of Section 3–5. But to emphasize the fact that the procedure will work equally well for any set of right-hand members, let us replace the right-hand member

$$\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \quad \text{by} \quad U = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

thus:

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{1}$$

This is an equation of the form

$$AX = U; \tag{2}$$

the coefficient matrix A determines a matrix function with independent variable X on the range of 3×1 matrices and dependent variable U on a domain also of 3×1 matrices.

If the matrix A *has an inverse* A^{-1}, *so that*

$$A^{-1} A = I = A A^{-1},$$

and if we can determine A^{-1}, then we can solve the equation (2) for X in terms of U by multiplying on the left by A^{-1}:

$$A^{-1} AX = A^{-1} U,$$

whence

$$X = A^{-1} U:$$

thus we have the inverse matrix function of the matrix function given by Equation (2). Our problem is to determine the matrix A^{-1} in case this matrix exists for our particular example.

For symmetry, let us write Equation (1) in the equivalent form

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \qquad (3)$$

with a coefficient matrix on both sides of the equation.

Looking at the left-hand coefficient matrix in Equation (3), we determine a matrix multiplier to adjust the first column, as follows:

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

This multiplies the first row of the matrix of coefficients by 1/3, adds $-2(1/3) = -2/3$ times the first row to the second, and adds $1(1/3) = 1/3$ times the first to the third, yielding

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & -\frac{7}{3} & -\frac{8}{3} \\ 0 & \frac{5}{3} & \frac{13}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 1 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

Adjusting the second column of the left-hand coefficient matrix, we have

$$\begin{bmatrix} 1 & \frac{2}{7} & 0 \\ 0 & -\frac{3}{7} & 0 \\ 0 & \frac{5}{7} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & -\frac{7}{3} & -\frac{8}{3} \\ 1 & \frac{5}{3} & \frac{13}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{7} & 0 \\ 0 & -\frac{3}{7} & 0 \\ 0 & \frac{5}{7} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

This multiplies the second row by $-3/7$, adds $(-2/3)(-3/7) = 2/7$ times the second row to the first, and adds $(-5/3)(-3/7) = 5/7$ times the second row to the third to yield numbers 0, 1, and 0 in the second column:

$$\begin{bmatrix} 1 & 0 & -\frac{10}{7} \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & \frac{17}{7} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} & 0 \\ \frac{2}{7} & -\frac{3}{7} & 0 \\ -\frac{1}{7} & \frac{5}{7} & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

Similarly, adjusting the third column of the left-hand coefficient matrix, we perform the multiplication

$$\begin{bmatrix} 1 & 0 & \frac{10}{17} \\ 0 & 1 & -\frac{8}{17} \\ 0 & 0 & \frac{7}{17} \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{10}{17} \\ 0 & 1 & \frac{8}{7} \\ 0 & 0 & \frac{17}{7} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{10}{17} \\ 0 & 1 & -\frac{8}{17} \\ 0 & 0 & \frac{7}{17} \end{bmatrix} \begin{bmatrix} \frac{1}{7} & \frac{2}{7} & 0 \\ \frac{2}{7} & -\frac{3}{7} & 0 \\ -\frac{1}{7} & \frac{5}{7} & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

obtaining

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{17} & \frac{12}{17} & \frac{10}{17} \\ \frac{6}{17} & -\frac{13}{17} & -\frac{8}{17} \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{4}$$

Equation (4) can be written equivalently as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{17} & \frac{12}{17} & \frac{10}{17} \\ \frac{6}{17} & -\frac{13}{17} & -\frac{8}{17} \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{5}$$

In particular, to return to our original system (I) of Section 3-5, if for

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \text{we take} \quad \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix},$$

we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{17} & \frac{12}{17} & \frac{10}{17} \\ \frac{6}{17} & -\frac{13}{17} & -\frac{8}{17} \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix},$$

which coincides with the result obtained in Section 3-4 by the triangulation method.

Again, for the problem

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix},$$

we obtain

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{17} & \frac{12}{17} & \frac{10}{17} \\ \frac{6}{17} & -\frac{13}{17} & -\frac{8}{17} \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{90}{17} \\ -\frac{106}{17} \\ \frac{63}{17} \end{bmatrix}$$

You should compare Equations (1) and (5). Let

$$A = \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{17} & \frac{12}{17} & \frac{10}{17} \\ \frac{6}{17} & -\frac{13}{17} & -\frac{8}{17} \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix},$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad U = \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

The above procedure amounts to multiplying

$$AX = U$$

on the left by B,

$$BAX = BU,$$

to obtain

$$X = BU.$$

Accordingly, we have

$$BA = I, \tag{6}$$

the identity matrix. You can show likewise that

$$AB = I. \tag{7}$$

Thus the matrix A has B as its inverse, as defined in Chapter 1:

$$B = A^{-1}.$$

Is it an accident that the matrix B, which we determined as the product of elementary matrices in such a way as to satisfy Equation (6), also satisfies Equation (7)? Not at all! — even though we know that the commutative law does not generally hold for matrix multiplication. The fact that Equation (7) follows from Equation (6) is an instance of the following result:

Theorem 3-1. If A and B are $n \times n$ matrices, if B is a product of elementary matrices, and if $BA = I$, where I is the $n \times n$ identity matrix, then

$$AB = I.$$

Proof. For simplicity, we shall give the proof only for the representative case

$$B = E_2 E_1,$$

where E_1 and E_2 are elementary matrices; and we shall use the fact that every elementary matrix E has an inverse E^{-1} as indicated on pages 132 and 133.

Since

$$BA = I \quad \text{and} \quad B = E_2 E_1,$$

we have

$$E_2 E_1 A = I,$$

whence

$$E_1^{-1} E_2^{-1} E_2 E_1 A = E_1^{-1} E_2^{-1} I.$$

Consequently,

$$E_1^{-1}(E_2^{-1}\ E_2)\ E_1\ A = E_1^{-1}\ E_2^{-1}\ I,$$

or

$$E_1^{-1}\ E_1\ A = E_1^{-1}\ E_2^{-1},$$

since $E_2^{-1}\ E_2 = I$ and the product of I by any matrix is the matrix itself. But also $E_1^{-1}\ E_1 = I$, so that

$$A = E_1^{-1}\ E_2^{-1}.$$

Since

$$A = E_1^{-1}\ E_2^{-1} \quad \text{and} \quad B = E_2\ E_1,$$

we have

$$\begin{aligned} AB &= E_1^{-1}\ E_2^{-1}\ E_2\ E_1 \\ &= E_1^{-1}(E_2^{-1}\ E_2)E_1, \end{aligned}$$

whence

$$AB = E_1^{-1}\ E_1,$$

or

$$AB = I,$$

as desired. This completes the proof of the theorem

Now look once more at the left-hand members and at the right-hand members of the equations starting with Equation (3) and ending with Equation (4), and suppress from them the matrices

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

Thus on the left and right, respectively, you start with

$$A \quad \text{and} \quad I$$

and end with

$$I \quad \text{and} \quad A^{-1}:$$

The sequence of transformation matrices that leads from A to I leads also from I to A^{-1}. We have thus outlined a method for the determination of the inverse of the matrix A.

Some matrices, however, do not have inverses, as you learned in Chapters 1 and 2. We shall be concerned with such a matrix when we deal with the matrix of coefficients in the examples (a) and (b) at the start of the next section.

Exercises 3–6

1. Solve the following systems of equations by the diagonal method:

$$\text{(a)} \quad \begin{aligned} 3x + 2y - 2z &= -4, \\ 2x - y - 4z &= 2, \\ -x + y + 5z &= 7; \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x - y - 2z &= 3, \\ y + 3z &= 5, \\ 2x + 2y - 3z &= 15, \end{aligned}$$

2. Solve by the diagonal method:

$$\text{(a)} \quad \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 6 \\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}, \qquad \text{(b)} \quad \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 6 \\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix},$$

$$\text{(c)} \quad \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 6 \\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}, \qquad \text{(d)} \quad \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 6 \\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

3. Solve by the diagonal method:

$$\begin{bmatrix} 4 & 0 & 2 \\ 1 & 3 & 1 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} x & u & m & r \\ y & v & n & s \\ z & w & p & t \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 & 10 \\ 1 & 6 & 2 & 6 \\ 3 & 7 & 7 & 12 \end{bmatrix}.$$

4. Solve by the diagonal method:

$$\begin{aligned} 2x + y + 2z - 3w &= 0, \\ 4x + y + z + w &= 15, \\ 6x - y - z - w &= 5, \\ 4x - 2y + 3z - w &= 2. \end{aligned}$$

5. Solve by the diagonal method:

$$\begin{aligned} 9x - y &= 37, \\ 8y - 2z &= -4, \\ 7z - 3w &= -17, \\ 2x + 6w &= 14. \end{aligned}$$

6. Determine the inverse of each of the following matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 5 \\ -2 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 4 & 0 \\ \frac{1}{2} & 3 & 2 \end{bmatrix}.$$

7. Use your work on Exercise 4 to solve

$$\begin{bmatrix} 2 & 1 & 2 & -3 \\ 4 & 1 & 1 & 1 \\ 6 & -1 & -1 & -1 \\ 4 & -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} s & w \\ t & x \\ u & y \\ v & z \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 6 & 12 \\ 4 & 8 \\ -2 & 7 \end{bmatrix}.$$

8. Explain how it is that the diagonal process is not self-destructive — that is, that after a 0 or 1 has been established in a certain position in the coefficient matrix, this value persists at that place in subsequent steps.

9. Express the matrix

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix}$$

as a product of elementary matrices.

10. Give a proof of Theorem 3-1 for the case

$$B = E_3 E_2 E_1,$$

where E_1, E_2, and E_3 are elementary matrices. Try to prove the theorem for the general case

$$B = E_n E_{n-1} \cdots E_2 E_1.$$

3-7. Linear Systems in General

Earlier in this chapter, in Exercise 3-2-3, we were concerned with the linear systems,

$$\text{(a)} \quad \begin{aligned} x + 2y - z &= 3, \\ x - y + z &= 4, \\ 4x - y + 2z &= 14; \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x + 2y - z &= 3, \\ x - y + z &= 4, \\ 4x - y + 2z &= 15, \end{aligned}$$

which look innocent enough. But for them the first step of the triangulation method of solution yields

$$\text{(a')} \quad \begin{aligned} x + 2y - z &= 3, \\ - 3y + 2z &= 1, \\ - 9y + 6z &= 2; \end{aligned} \qquad \text{(b')} \quad \begin{aligned} x + 2y - z &= 3, \\ - 3y + 2z &= 1, \\ - 9y + 6z &= 3, \end{aligned}$$

and the second step gives

$$\text{(a'')}\quad \begin{aligned} x + 2y - z &= 3, \\ -3y + 2z &= 1, \\ 0 &= -1; \end{aligned} \qquad\qquad \text{(b'')}\quad \begin{aligned} x + 2y - z &= 3, \\ -3y + 2z &= 1, \\ 0 &= 0. \end{aligned}$$

If there were a solution for the system (a), then we would have $0 = -1$; hence there is no solution for this system.

Now, by contrast, there is no mathematical loss in dropping the equation $0 = 0$ from the system (b''). Without this equation, the system can be written equivalently as

$$\text{(b''')}\quad \begin{aligned} x + 2y &= 3 + z, \\ y &= -\frac{1}{3} - \frac{2}{3}z. \end{aligned}$$

Application of the backward-solution portion of the triangulation method to the system (b''') yields

$$\begin{aligned} x &= \frac{11}{3} + \frac{7}{3}z, \\ y &= \frac{1}{3} - \frac{2}{3}z. \end{aligned} \tag{1}$$

Whatever value is given to z, this value and the corresponding values of x and y as determined by the equations (1) satisfy the original system (b). For example, a few solutions are shown in the following table:

z	x	y
−2	−1	1
−1	$\frac{4}{3}$	$\frac{1}{3}$
0	$\frac{11}{3}$	$-\frac{1}{3}$
1	6	−1
4	13	−3

Do you have an intuitive geometric notion of what might be going on in the above systems (a) and (b)? Relative to a 3-dimensional rectangular coordinate system, each of the 3 equations in either (a) or (b) represents a plane. Each pair of planes actually intersect in a line. The 3 lines of intersection in either (a) or (b) might be expected to be concurrent in a point. However, in (a) the 3 lines are parallel but not coincident; there is no point that lies on all 3 planes. On the other hand, in (b) the 3 lines are parallel and coincident; there is an entire "line" of solutions.

How many possible configurations, as regards intersections, can you list for 3 planes, not necessarily distinct from one another? They might, for example, have exactly one point in common; or two might be coincident and the third distinct from but parallel to them; and so on. There are systems of linear equations that correspond to each of these geometric situations.

Here are two additional systems that even more obviously than the above system (a) have no solutions:

$$\text{(c)}\quad \begin{aligned} x &= 2, \\ x &= 3; \end{aligned} \qquad\qquad \text{(d)}\quad \begin{aligned} x + y + z &= 2, \\ x + y + z &= 3. \end{aligned}$$

Thus you see that the number of variables as compared with the number of equations does not determine whether or not there is a solution.

It is plain that the routine triangulation and diagonal methods can be applied to systems of any number of linear equations in any number of variables. Let us examine the general situation and see what can happen. Suppose we have a system of linear equations in certain variables. If any variable occurs with coefficient 0 in every equation, it plays no role and we drop it. Suppose we have applied k molecules of the diagonal process; in doing this, since we sometimes divide by the coefficient of one of the variables, it might be necessary to rearrange some of the equations, by the method of paragraph (c') in Section 3-4, or to rearrange some of the terms in all the equations. At the end of this process, we arrive at an equivalent set of equations of the form

$$\left.\begin{array}{l}
x_1 + \qquad\qquad \text{linear terms in variables other than } x_1,\ldots,x_k = b_1,\\
\qquad x_2 + \qquad \text{linear terms in variables other than } x_1,\ldots,x_k = b_2,\\
\qquad\qquad\qquad\qquad\qquad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
\qquad\qquad x_k + \text{linear terms in variables other than } x_1,\ldots,x_k = b_k,\\
\text{and OTHER EQUATIONS in which the variables } x_1,\ldots,x_k \text{ do not appear.}
\end{array}\right\}(2)$$

If any variable occurs with a nonzero coefficient in any one of the OTHER EQUATIONS, we can continue our elimination process. Eventually we will come to an end. At this point, our system of equations must look like this:

$$\left.\begin{array}{l}
x_1 + \qquad\qquad \text{linear terms in variables other than } x_1,\ldots,x_k = b_1,\\
\qquad x_2 + \qquad \text{linear terms in variables other than } x_1,\ldots,x_k = b_2,\\
\qquad\qquad\qquad\qquad\qquad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
\qquad\qquad x_k + \text{linear terms in variables other than } x_1,\ldots,x_k = b_k,\\
\text{OTHER EQUATIONS in which no variable appears with a nonzero coefficient.}
\end{array}\right\}(3)$$

What can one of these OTHER EQUATIONS, which must be an equation in which no variable appears, look like? Either it is of the form $0 = 0$, in which case we might as well drop it; or it is of the form $0 = b$, where b is a constant different from zero, in which case it is a contradiction. Hence we see that: either the OTHER EQUATIONS all state simply $0 = 0$, in which case they can be dropped, or at least one of the OTHER EQUATIONS is an obvious contradiction. Since the system of equations (3) is equivalent to the original system of equations, (3) can contain contradictions, i.e., state impossibilities, only if the original system of equations also states impossibilities, i.e., only if the original system of equations simply has no solution.

Thus, if the OTHER EQUATIONS in (3) are not all simply $0 = 0$, perhaps repeated several times, then the original system of equations has no solution.

Summarizing, we see that we have established the following result:

Theorem 3-2. If the diagonal method described above is repeatedly applied to an arbitrary system of linear equations and carried through to the end, then we arrive at one of these two situations:

(a) at least one palpable contradiction of the form $0 = b$, b being some nonzero number, so that the orginal system of equations has no solution;

(b) an equivalent system of equations of the form

$$\left.\begin{array}{lll} x_1 + & \text{linear terms in variable other than } x_1,\ldots,x_k & = b_1, \\ \quad x_2 + & \text{linear terms in variables other than } x_1,\ldots,x_k & = b_2, \\ & \cdots\cdots & \\ \qquad x_k + & \text{linear terms in variables other than } x_1,\ldots,x_k & = b_k. \end{array}\right\} \quad (4)$$

Let us examine case (b) more carefully. There are two subcases: either (i) there are really no variables other than $x_1,\ldots,x_k$; or (ii) there really are variables other than $x_1,\ldots,x_k$.

In case (i) our system of equations reduces to $x_1 = b_1$, $x_2 = b_2,\ldots,$ $x_k = b_k$, and the solution is unique.

In case (ii), there are variables other than $x_1,\ldots,x_k$. Denote the variables other than $x_1,\ldots,x_k$ by the letters $y_1,y_2,\ldots,y_n$, where $n \geq 1$. We can transpose and write the system (4) of equations in the form

$$\left.\begin{array}{l} x_1 = b_1 + c_1y_1 + d_1y_2 + \cdots + e_1y_u \\ x_2 = b_2 + c_2y_1 + d_2y_2 + \cdots + e_2y_n \\ \quad \cdots\cdots \\ x_k = b_k + c_ky_1 + d_ky_2 + \cdots + e_ky_n \end{array}\right\} \quad (5)$$

It is clear that this system of equations will be satisfied if we assign arbitrary values to the variables $y_1,\ldots,y_n$, and then determine the values of $x_1,\ldots,x_k$ from (5). In this case, our solution evidently is not unique, as was illustrated in the table on page 152.

Summarizing, we have the following theorem:

Theorem 3-3. Let a system of arbitrarily many linear equations in arbitrarily many variables be given. If the diagonal method is repeatedly applied to the given system of linear equations, and carried through to the end, then we arrive at one of these three situations:

(a) at least one palpable contradiction of the form $0 = b$, b being some nonzero number, so that the original system of equations has no solution;

(b) an equivalent system of the form $x_1 = b_1$, $x_2 = b_2, \dots, x_k = b_k$, one for each of the unknowns in the original system of equations, so that there is a unique solution;

(c) an equivalent system of the form

$$\left.\begin{aligned} x_1 &= b_1 + c_1y_1 + d_1y_2 + \cdots + e_1y_n, \\ x_2 &= b_2 + c_2y_1 + d_2y_2 + \cdots + e_2y_n, \\ &\quad . \quad . \quad . \quad . \quad . \\ x_k &= b_k + c_ky_1 + d_ky_2 + \cdots + e_ky_n, \end{aligned}\right\} \qquad (6)$$

the unknowns of the initial system being $x_1, \dots, x_k$ and $y_1, \dots, y_n$, and not all coefficients of the y's different from 0, so that there is an infinitude of solutions, which are obtained by giving arbitrary values to the variables $y_1, \dots, y_n$ and then determining the remaining variables $x_1, \dots, x_k$ from the equations (6).

Thus the question of solving systems of linear equations in arbitrarily many unknowns is settled in all possible cases.

Exercises 3-8

1. (a) List all possible configurations, as regards intersections, for 3 distinct planes.

   (b) List also the additional possible configurations if the planes are allowed to be coincident.

2. Solve by the diagonal process:

$$\begin{aligned} x + y + z &= 6, \\ x + y + 2z &= 7, \\ y + z &= 1. \end{aligned}$$

3. Find the solutions, if any, of the system of equations

$$\begin{aligned} 2v + x + y + z &= 0, \\ v - x + 2y + z &= 0, \\ 4v - x + 5y + 3z &= 1, \\ v - x + y - z &= 2. \end{aligned}$$

4. Find the solutions, if any, of the system of equations

$$\begin{aligned} x + y + z &= 1, \\ x - y - 2z &= 0, \\ x + 2y + 3z &= 1, \\ 3x - y - 5z &= 1. \end{aligned}$$

5. Find the solutions, if any, of the system of equations

$$\begin{aligned} v + 2x + y + z &= 0, \\ -v + x + 2y + z &= 0, \\ -v + 4x + 5y + 3z &= 1. \end{aligned}$$

6. Find the solutions, if any, of the system of equations

$$\begin{aligned} v + x + y + z &= 1, \\ v - x - y + z &= 2, \\ v + 2x - y + 2z &= 0, \\ v - 3x - 3y - 7z &= 4. \end{aligned}$$

Chapter 4

REPRESENTATION OF COLUMN MATRICES AS GEOMETRIC VECTORS

4-1. The Algebra of Vectors

In the present chapter, we shall develop a simple geometric representation for a special class of matrices — namely, the set of column matrices $\begin{bmatrix} a \\ b \end{bmatrix}$ with two entries each. The familiar algebraic operations on this set of matrices will be reviewed and also given geometric interpretation, which will lead to a deeper understanding of the meaning and implications of the algebraic concepts.

By definition, *a column vector of order* 2 is a 2×1 matrix. Consequently, using the rules of Chapter 1, you can add two such vectors or multiply any one of them by a number. The set of column vectors of order 2 has, in fact, an algebraic structure whose properties were largely explored in your study of the rules of operation with matrices. In the following pair of theorems, we summarize what you already know of the algebra of these vectors, and in the next section we shall begin the interpretation of that algebra in geometric terms.

Theorem 4-1. Let V and W be column vectors of order 2 and let A be a square matrix of order 2. Let r be a number. Then

$$V + W, \quad rV, \quad \text{and} \quad AV$$

are each column vectors of order 2.

Theorem 4-2. Let V, W, and U be column vectors of order 2, and let A and B be square matrices of order 2. Let r and s be numbers. Then all the following laws are valid.

I. Laws for the addition of vectors :

(a) $V + W = W + V$,

(b) $(V + W) + U = V + (W + U)$,

(c) $V + 0_{2\times 1} = V$,

(d) $V + (-V) = 0_{2\times 1}$.

II. Laws for the numerical multiplication of vectors:

(a) $r(V + W) = rV + rW$,

(b) $r(sV) = (rs)V$,

(c) $(r + s)V = rV + sV$,

(d) $0V = 0_{2\times 1}$,

(e) $1V = V$,

(f) $r0_{2\times 1} = 0_{2\times 1}$.

III. Laws for the multiplication of vectors by matrices:

(a) $A(V + W) = AV + AW$,

(b) $(A + B)V = AV + BV$,

(c) $A(BV) = (AB)V$,

(d) $0_2 V = 0_{2\times 1}$,

(e) $IV = V$,

(f) $A(rV) = (rA)V = r(AV)$.

In reading Theorem 4–2, recall that $0_{2\times 1}$ is the column vector of order 2, and 0_2 the square matrix of order 2, all of whose entries are 0.

Both of the preceding theorems have already been proved for matrices. Since vectors are merely special types of matrices, the theorems as stated must likewise be true. They would also be true, of course, if 2 were replaced by 3, or by a general n, throughout, with the understanding that a column vector of order n is a matrix of order n × 1.

Exercises 4-1

1. Let

$$V = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad W = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} -4 \\ 2 \end{bmatrix};$$

let

$$A = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}; \quad \text{and}$$

let $r = 2$ and $s = -1$. Verify each of the laws stated in Theorem 4-2 for this choice of values for the variables.

2. Determine the vector V such that $AV - AW = AW + BW$, where

$$A = \begin{bmatrix} 5 & 1 \\ 4 & -2 \end{bmatrix}, \quad W = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}.$$

3. Determine the vector V such that $2V + 2W = AV + BV$, if

$$W = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix}.$$

4. Find V, if

$$A \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \text{ and } \quad A(3V) = A(BV).$$

5. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Evaluate

(a) $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and (b) $A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) Using your answers to parts (a) and (b), determine the entries of A if, for every vector V of order 2,

$$AV = 0_{2\times 1}.$$

(d) State your result as a theorem.

6. Restate the theorem obtained in Exercise 5 if A is a square matrix of order n and V stands for any column vector of order n. Prove the new theorem for $n = 3$. Try to prove the theorem for all n.

7. Using your answers to parts (a) and (b) of Exercise 5, determine the entries of A if, for every vector V of order 2,

$$AV = V.$$

State your result as a theorem.

8. Restate the theorem obtained in Exercise 7 if A is a square matrix of order n and V stands for any column vector of order n. Prove this theorem for $n = 3$. Try to prove the theorem for all n.

9. Theorems 4-1 and 4-2 summarize the properties of the algebra of column vectors with n entries. State two analogous theorems summarizing the properties of the algebra of row vectors with n entries. Show that the two algebraic structures are isomorphic.

4-2. Vectors and Directed Line Segments

In graphing functions and relationships, you discovered the great advantage in having a simple numerical language to describe the location of a point in a plane. You remember that an ordered pair of real numbers constitutes the coordinates of any given point in the plane. But that same ordered pair of numbers can be regarded as a row vector or as a column vector.

Thus, in Figure 4-1 the point P that is 3 units to the left of the y axis and 4 units above the x axis is represented by the pair of numbers $(-3,4)$. However, that same number couple, written $\begin{bmatrix} -3 & 4 \end{bmatrix}$, is simply a row vector; written $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$, it is a column vector. Consequently, a row or column vector with two entries (or components), $\begin{bmatrix} u & v \end{bmatrix}$ or $\begin{bmatrix} u \\ v \end{bmatrix}$, can be represented geometrically by the point $P:(u,v)$ in a given rectangular coordinate plane.

It is often more useful, though, to think of the row or column vector as being represented by the directed line segment from 0 to P. We denote this directed line segment by the symbol: $\overrightarrow{OP}$. Thus, the row or column vector is represented by a geometric quantity having length and direction. We shall call this geometric quantity a geometric vector.

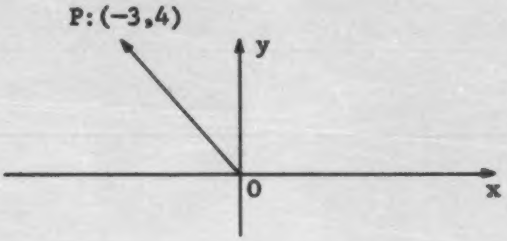


Figure 4-1. A geometric vector.

In Figure 4-1, the directed line segment or geometric vector $\overrightarrow{OP}$ is pictured by the arrow drawn from 0 to P.

The length of $\overrightarrow{OP}$ is easily calculated by using the Pythagorean Theorem. For the point P: (-3,4), the length of $\overrightarrow{OP}$ is

$$\sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = 5.$$

One way of specifying the direction of $\overrightarrow{OP}$ is simply to say that its direction is that of the ray issuing from the origin and passing through (-3,4). It is much more useful, however, to indicate the direction of the ray by giving the cosine and sine of the angle having the ray as terminal side and the positive x axis as initial side. Thus, the direction is specified by the numbers -3/5 and 4/5. You can verify the correctness of these numbers by recalling that the cosine and sine of an angle in standard position, that is, an angle placed in the coordinate plane so that its initial side is the positive x axis and its terminal side is the ray that issues from the origin and passes through another point (x_1, y_1), are given by the respective formulas

$$\frac{x_1}{\sqrt{x_1^2 + y_1^2}} \quad \text{and} \quad \frac{y_1}{\sqrt{x_1^2 + y_1^2}} \; .$$

Regarding these numbers -3/5 and 4/5, it is worth while noticing that -3/5 is the cosine of the angle that $\overrightarrow{OP}$ forms with the positive x axis and 4/5 is the cosine of the angle that $\overrightarrow{OP}$ forms with the positive y axis; consequently, these numbers are called the "direction cosines" of $\overrightarrow{OP}$. (Can you tell why the slope of $\overrightarrow{OP}$, i.e., the number -4/3, will not specify the direction of the line segment from 0 to P?)

In general, the column vector

$$V = \begin{bmatrix} u \\ v \end{bmatrix}$$

is represented by the directed line segment $\overrightarrow{OP}$ from the origin to the point P:(u,v). The length of $\overrightarrow{OP}$ is called the length or the norm of V. Using the symbol $||V||$ to stand for the norm of V, we have

$$||V|| = \sqrt{u^2 + v^2} \; .$$

Thus, if not both u and v are zero, the direction cosines of $\overrightarrow{OP}$ are

$$\frac{u}{||V||} \quad \text{and} \quad \frac{u}{||V||} \, ,$$

respectively.

Similar statements could be made concerning the row vector $\begin{bmatrix} u, v \end{bmatrix}$, but for the present we shall consider only column vectors and the corresponding geometric vectors. Hereafter, the term "vector" will be used to mean "column vector."

We shall call $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ the zero vector or the null vector. It will be regarded as being represented by a geometric vector of length zero to which no unique direction is assigned. For the sake of convenience, however, we shall say that the zero vector is directed and that it has the same direction as any and every other vector.

Consequently, each vector $\begin{bmatrix} u \\ v \end{bmatrix}$ determines a unique directed line segment issuing from the origin of a given rectangular coordinate system. Conversely, each such directed line segment determines a unique vector $\begin{bmatrix} u \\ v \end{bmatrix}$. Thus, a one-to-one correspondence has been set up between the set of column vectors having two real-number entries and the set of directed line segments lying in a Cartesian coordinate plane and issuing from the origin. In the next section, we shall discover an interpretation of the algebraic operations on vectors in terms of geometric operations on directed line segments.

The association between vectors and directed line segments introduced in this section is as applicable to 3-dimensional space as it is to the 2-dimensional plane. The only difference is that a directed line segment in 3-dimensional space will represent a vector of order 3, not a vector of order 2.

Exercises 4-2

1. Of the following pairs of vectors,

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}$;

(b) $\begin{bmatrix} 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \end{bmatrix}$;

(c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix}$;

(d) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;

(e) $\begin{bmatrix} -9 \\ -2 \end{bmatrix}, \begin{bmatrix} 2\sqrt{15} \\ -5 \end{bmatrix}$

(f) $\begin{bmatrix} -5 \\ -12 \end{bmatrix}, \begin{bmatrix} 12 \\ 5 \end{bmatrix}$;

(g) $\begin{bmatrix} \sqrt{2} \\ -3\sqrt{2} \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \end{bmatrix}$;

(h) $\begin{bmatrix} -8 \\ 15 \end{bmatrix}, \begin{bmatrix} 16 \\ 30 \end{bmatrix}$;

(i) $\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix}$;

(j) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3t \\ 4t \end{bmatrix}$,

which have the same length? Which have the same direction?

2. Let $V = t\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Draw the arrows representing V for

$t = 1$, $t = 2$, $t = 3$, $t = -1$, $t = -2$, and $t = -3$.

In each case, compute the length and direction cosines of V.

3. In a rectangular coordinate plane, draw the directed line segments representing the members of each of the following sets of vectors. Use a different coordinate plane for each set of vectors. Find the length and direction cosines of each vector:

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$;

(b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$;

(c) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$;

(d) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$;

(e) $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ -4 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

4. Let $V = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Draw the line segments representing V for $x = 1$, $x = 2$, $x = 3$, $x = -1$, $x = -2$, and $x = -3$. In each case, compute the length and direction cosines of V.

5. Let $V = \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ m \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix}$.

Draw the line segments representing V, if $t = 0, \pm 1, \pm 2$, and

(a) $m = 1$, $b = 0$;

(b) $m = 2$, $b = 1$;

(c) $m = -1/2$, $b = 3$.

In each case, verify that the corresponding set of five points (x,y) lies on a line.

6. Two vectors are called collinear provided the geometric vectors representing them lie on the same line through the origin. If A and B are nonzero

collinear vectors, determine the two possible relationships between the direction cosines of A and the direction cosines of B.

7. Determine all the vectors of the form $\begin{bmatrix} u \\ v \end{bmatrix}$ that are collinear with

(a) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$,

(b) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$,

(c) $\begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 10 \end{bmatrix}$,

(d) $\begin{bmatrix} 9 \\ -5 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}$,

(e) $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

4-3. Geometrical Interpretation of the Multiplication of a Vector by a Number

The geometrical significance of the multiplication of a vector by a number is readily guessed on comparing the geometrical representations of the vectors V, 2V, and −2V for

$$V = \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

By definition,

$$2V = \begin{bmatrix} -6 \\ 8 \end{bmatrix},$$

while

$$-2V = \begin{bmatrix} 6 \\ -8 \end{bmatrix}.$$

Thus, as you can see in Figures 4-2 and 4-3, the arrows representing V and 2V have the same direction, while −2V is represented by an arrow pointing in the opposite direction. The length of the arrow associated with V is 5, while

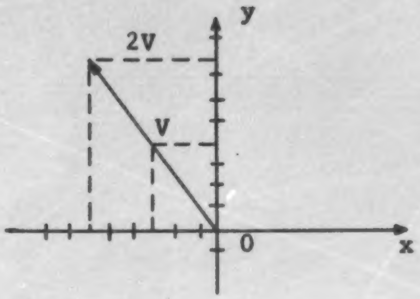


Figure 4-2. The product of
a vector and a positive number.

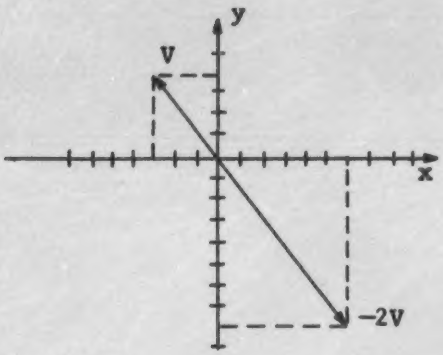


Figure 4-3. The product of
a vector and a negative number.

the arrows representing $2V$ and $-2V$ each have length 10. Thus, multiplying V by 2 produced a stretching of the associated geometric vector to twice its original length while leaving its direction unchanged. Multiplication by -2 not only doubled the length of the arrow but also reversed its direction.

These observations lead us to formulate the following theorem.

Theorem 4-3. Let the directed line segment $\overrightarrow{OP}$ represent the vector V and let r be a number. Then the vector rV is represented by a directed line segment whose length is $|r|$ times the length of $\overrightarrow{OP}$. If $r \geq 0$, the representative of rV has the same direction as $\overrightarrow{OP}$; if $r < 0$, the direction of the representative of rV is opposite to that of $\overrightarrow{OP}$.

Proof. Let V be the vector $\begin{bmatrix} u \\ v \end{bmatrix}$. Then

$$||V|| = \sqrt{u^2 + v^2} .$$

Now,

$$rV = \begin{bmatrix} ru \\ rv \end{bmatrix} ;$$

hence,

$$
\begin{aligned}
||rV|| &= \sqrt{(ru)^2 + (rv)^2} \\
&= \sqrt{r^2(u^2\ v^2)} \\
&= |r|\sqrt{u^2 + v^2} \\
&= |r|\ \ ||V||.
\end{aligned}
$$

This proves the first part of the theorem.

If

$$r = 0 \quad \text{or} \quad V = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

the second part of the theorem is certainly true.

If

$$r \neq 0 \quad \text{and} \quad V \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

the direction cosines of $\overrightarrow{OP}$ are

$$\frac{u}{||V||} \quad \text{and} \quad \frac{v}{||V||},$$

while those of the representative of rV are

$$\frac{ru}{|r|\ \ ||V||} \quad \text{and} \quad \frac{rv}{|r|\ \ ||V||},$$

If $r > 0$, we have $|r| = r$, whence it follows that the arrows associated with V and rV have the same direction cosines and, therefore, the same direction. If $r < 0$, we have $|r| = -r$, and the direction cosines of the arrow associated with rV are the negatives of those of $\overrightarrow{OP}$. Thus, the direction of the representative of rV is opposite to that of $\overrightarrow{OP}$. This completes the proof of the theorem.

One way of stating part of the theorem just proved is to say that if r is a number and V is a vector, then V and rV are collinear vectors; that is, they are represented by arrows lying on the same line through the origin. On the other hand, if the arrows representing two vectors are collinear, it is easy to show that you can always express one of the vectors as the product of the other vector by a suitably chosen number. Thus, by checking direction cosines, it is easy to verify that

$$\begin{bmatrix}5\\-2\end{bmatrix}, \quad \begin{bmatrix}50\\-20\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}-10\\4\end{bmatrix}$$

are collinear vectors, and that

$$\begin{bmatrix}50\\-20\end{bmatrix} = 10\begin{bmatrix}5\\-2\end{bmatrix}, \qquad \text{while} \qquad \begin{bmatrix}-10\\4\end{bmatrix} = -2\begin{bmatrix}5\\-2\end{bmatrix}.$$

In the exercises that follow, you will be asked to show why the general result illustrated by this example holds true.

Exercises 4-3

1. Let L be the set of all vectors collinear with the vector $\begin{bmatrix}2\\3\end{bmatrix}$. Fill in the following blanks so as to produce in each case a true statement:

   (a) $\begin{bmatrix}4\\-\end{bmatrix} \in L$;

   (b) $\begin{bmatrix}-\\9\end{bmatrix} \in L$;

   (c) $\begin{bmatrix}-2/3\\-\end{bmatrix} \in L$;

   (e) $\begin{bmatrix}6\\-\end{bmatrix} \notin L$;

   (f) for every real number t, $\begin{bmatrix}8t\\-\end{bmatrix} \in L$;

   (g) for every real number t, $\begin{bmatrix}-\\-12t\end{bmatrix} \in L$;

   (h) for every real number $h \neq 0$, $\begin{bmatrix}h\\-\end{bmatrix} \notin L$.

3. Verify graphically and prove algebraically that the vectors in each of the following pairs are collinear. In each case, express the first vector as the product of the second vector by a number:

(a) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; (d) $\begin{bmatrix} 2 \\ -32 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \end{bmatrix}$;

(b) $\begin{bmatrix} 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 10 \\ 8 \end{bmatrix}$; (e) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -8 \\ 4 \end{bmatrix}$;

(c) $\begin{bmatrix} -12 \\ 15 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \end{bmatrix}$; (f) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \end{bmatrix}$.

4. Let V be a vector and W a nonzero vector such that V and W are collinear. Prove that there exists a real number r such that

$$V = rW.$$

5. Prove:

(a) If $rV = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $r \neq 0$, then $V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(b) If $rV = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $V \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $r = 0$.

6. Show that the vector $V + rV$ has the same direction as V if $r \geq -1$, and the opposite direction to V if $r < -1$. Show also that

$$\|V + rV\| = \|V\| \; |1 + r|.$$

4-4. Geometrical Interpretation of the Addition of Two Vectors

The addition of two vectors has a geometric interpretation that is somewhat less obvious than that for the multiplication of a vector by a number.

If

$$V = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{then} \quad V + W = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is evident from Figure 4–4 that $V + W$ is represented by the diagonal drawn from the origin to the fourth vertex of the parallelogram (actually, the rectangle), three of whose vertices are (1, 0), (0, 0), and (0, 1).

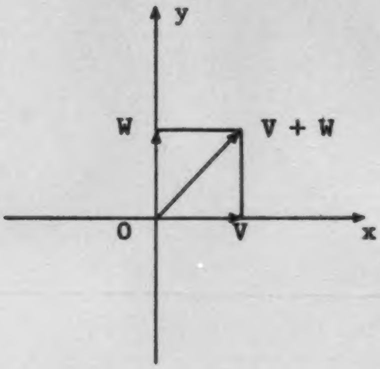


Figure 4–4. The addition of the vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

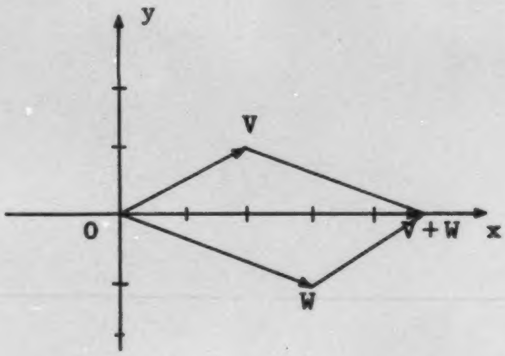


Figure 4–5. The addition of the vectors $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}3\\-1\end{bmatrix}$.

If

$$V = \begin{bmatrix}2\\1\end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix}3\\-1\end{bmatrix}, \quad \text{then} \quad V + W = \begin{bmatrix}5\\0\end{bmatrix}.$$

Looking at the representations of these three vectors in Figure 4–5, we see that $V + W$ this time is represented by the diagonal drawn from the origin to the fourth vertex of the parallelogram having (3, –1), (0, 0), and (2, 1) as three of its vertices.

The pattern common to our two examples certainly suggests that the addition of vectors corresponds to a kind of parallelogram rule for adding directed line segments. If the pattern holds in general, then $V + W$ is represented by the diagonal from the origin in the parallelogram having, as adjacent sides, the geometric vectors representing V and W.

A simple way to construct that diagonal is indicated in Figure 4–6. If $\overrightarrow{OP}$ represents V and $\overrightarrow{OT}$ represents W, construct the line segment PR

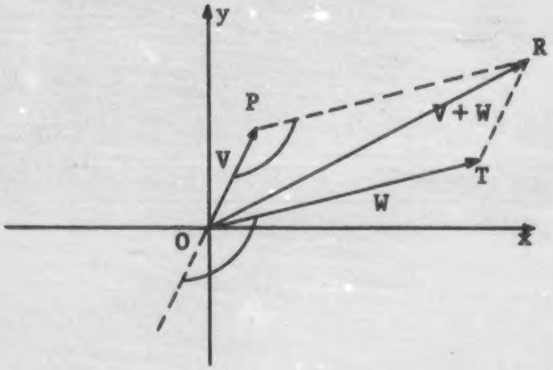


Figure 4–6. Construction for vector addition.

having the same length and direction as $\overrightarrow{OT}$. Then $\overrightarrow{OR}$ represents $V + W$.

This method of constructing the representative of $V + W$ has the advantage of being applicable even when V and W are collinear vectors. Our only problem is to verify that the construction actually does yield the representative of $V + W$, whatever choice is made for V and W. Let

$$V = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} r \\ s \end{bmatrix}.$$

Then

$$V + W = \begin{bmatrix} u + r \\ v + s \end{bmatrix}.$$

If V and W are not collinear, the points $(0, 0)$, (u, v) and (r, s) are distinct and constitute the vertices of a parallelogram; see Figure 4–7. To show that the fourth vertex of that parallelogram is the point $(u + r, v + s)$, you need merely show that the line segment joing $(0, 0)$ and (r, s) has the same length as the line segment joing (u, v) and $(u + r, v + s)$, and similarly that the line segment joing $(0,0)$ and (u, v) has the same length as the line segment joining (r, s) and $(u + r, v + s)$. Completing this argument is just an exercise

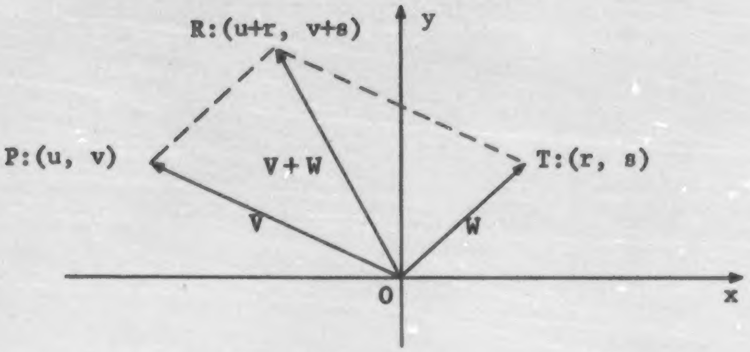


Figure 4-7. The addition of noncollinear vectors, $\begin{bmatrix} u \\ v \end{bmatrix}$ and $\begin{bmatrix} r \\ s \end{bmatrix}$.

in using the distance formula.

If V and W are collinear, the construction of the proposed representative of V + W is indicated in Figures 4-8 and 4-9. It is easy to verify that in

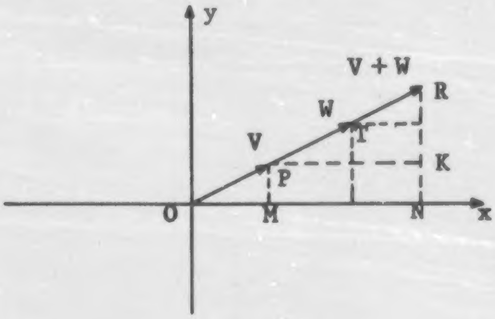


Figure 4-8. The addition of collinear vectors in the same direction.

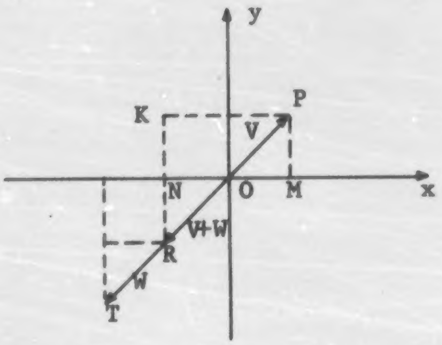


Figure 4-9. The addition of collinear vectors in opposite directions.

both cases we have the algebraic equalities

$$\begin{aligned} ON &= OM + MN \\ &= u + r \end{aligned}$$

and

$$NR = NK + KR$$
$$= v + s.$$

The details of the proof will be left as an exercise for the student.

We state the result formally as a theorem.

Theorem 4-4. If the vectors V and W are represented by the directed line segments $\overrightarrow{OP}$ and $\overrightarrow{OT}$, respectively, then $V + W$ is represented by $\overrightarrow{OR}$, where PR is the directed line segment having the same length and direction as $\overrightarrow{OT}$.

Since $V - W = V + (-W)$, the operation of subtracting one vector from another offers no essentially new geometric idea. Figure 4-10 illustrates the construction of the geometric vector representing $V - W$. It is useful to note, however, that since

$$||V - W|| = \left|\left|\begin{bmatrix} u - r \\ v - s \end{bmatrix}\right|\right| = \sqrt{(u - r)^2 + (v - s)^2},$$

the length of the vector $V - W$ equals the distance between the points P: (u, v) and T: (r, s).

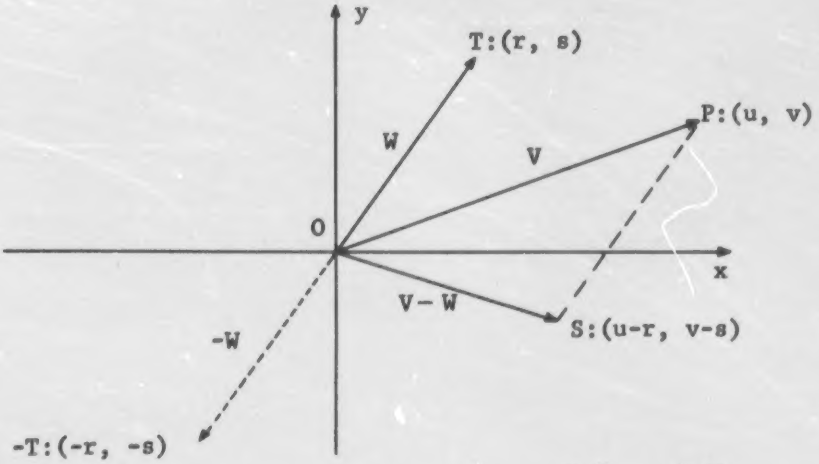


Figure 4-10. The subtraction of vectors, $\begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} r \\ s \end{bmatrix}$.

Exercises 4-4

1. Determine graphically the sum and differences of the following pairs of vectors. Does order matter in constructing the sum? the difference?

(a) $\begin{bmatrix}2\\4\end{bmatrix}, \quad \begin{bmatrix}7\\1\end{bmatrix};$ (d) $\begin{bmatrix}3\\6\end{bmatrix}, \quad 2\begin{bmatrix}-1\\2\end{bmatrix};$

(b) $\begin{bmatrix}3\\0\end{bmatrix}, \quad \begin{bmatrix}-5\\2\end{bmatrix};$ (e) $\begin{bmatrix}7\\4\end{bmatrix}, \quad \begin{bmatrix}-2\\-3\end{bmatrix};$

(c) $\begin{bmatrix}7\\2\end{bmatrix}, \quad \begin{bmatrix}-2\\-3\end{bmatrix};$ (f) $\begin{bmatrix}3\\-2\end{bmatrix}, \quad \begin{bmatrix}1\\0\end{bmatrix}.$

2. Illustrate graphically the associative law:

$$(V + W) + U = V + (W + U).$$

3. Compute each of the following graphically:

(a) $\begin{bmatrix}3\\7\end{bmatrix} + \begin{bmatrix}2\\-1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix},$

(b) $\begin{bmatrix}4\\-2\end{bmatrix} + \begin{bmatrix}-3\\2\end{bmatrix} + \begin{bmatrix}-1\\0\end{bmatrix},$

(c) $\begin{bmatrix}5\\2\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}4\\1\end{bmatrix} + \begin{bmatrix}-2\\-3\end{bmatrix},$

(d) $\begin{bmatrix}-1\\1\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}1\\5\end{bmatrix}.$

4. State the geometric significance of the following equations:

(a) $V + W = \begin{bmatrix}0\\0\end{bmatrix},$

(b) $V + W + U = \begin{bmatrix}0\\0\end{bmatrix},$

(c) $V + W + U + T = \begin{bmatrix}0\\0\end{bmatrix}.$

5. Complete the proof of both parts of Theorem 4-4.

4-5. The Inner Product of Two Vectors

Thus far in our development, we have investigated a geometrical interpretation for the algebra of vectors. We have established a one-to-one correspondence between the set of column vectors having two entries and the set of directed line segments from the origin of a coordinate plane. The algebraic operations of addition of two vectors and of multiplication of a vector by a number have acquired geometrical significance.

But we can also reverse our point of view and see that the geometry of vectors can lead us to the consideration of additional algebraic structure.

For instance, if you look at the pair of arrows drawn in Figure 4-11, you will very likely comment that they appear to be mutually perpendicular. You have begun to talk about the angle between the pair of arrows.

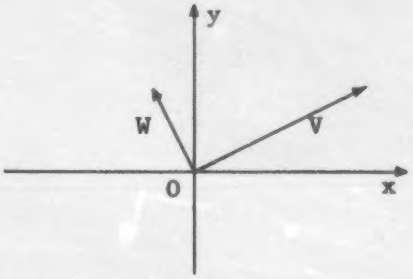


Figure 4-11. Perpendicular vectors.

Let us suppose, in general, that the points P, with coordinates (a,b), and R, with coordinates (c,d), are the terminal points of two geometric vectors. Consider the angle POR, which we denote by the Greek letter θ (theta), in the triangle POR of Figure 4-12.

It is very easy to compute the cosine of θ by applying the law of cosines to the triangle POR. If $|OP|$, $|OR|$, and $|PR|$, are the lengths of the sides of the triangle, then by the law of cosines we have

$$2|OP|\,|OR|\cos\theta = |OP|^2 + |OR|^2 - |PR|^2.$$

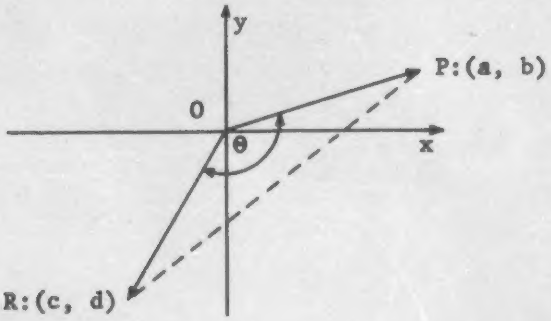


Figure 4–12. The angle between two vectors.

But

$$|OP| = \sqrt{a^2 + b^2} ,$$

$$|OR| = \sqrt{c^2 + d^2} ,$$

$$|PR| = \sqrt{(a-c)^2 + (b-d)^2} .$$

Thus,

$$2(\sqrt{a^2 + b^2})(\sqrt{c^2 + d^2}) \cos \theta = (a^2 + b^2) + (c^2 + d^2) - [(a-c)^2 + (b-d)^2]$$
$$= 2(ac + bd).$$

Hence,

$$|OP|\ |OR| \cos \theta = ac + bd. \qquad (1)$$

The number on the right–hand side of this equation, although clearly a function of the two vectors, has not heretofore appeared explicitly. Let us give it a name and introduce, thereby, a new binary operation for vectors.

Definition 4–2. The inner product of the vectors

$$\begin{bmatrix} a \\ b \end{bmatrix} \text{ and } \begin{bmatrix} c \\ d \end{bmatrix}, \text{ written } \begin{bmatrix} a \\ b \end{bmatrix} \bullet \begin{bmatrix} c \\ d \end{bmatrix},$$

is the algebraic sum of the products of corresponding entries. Symbolically,

$$\begin{bmatrix} a \\ b \end{bmatrix} \bullet \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd.$$

We can similarly define the inner product of two row vectors: $\begin{bmatrix} a & b \end{bmatrix} \bullet \begin{bmatrix} c & d \end{bmatrix} = ac + bd$.

Another name for the inner product of two vectors is the "dot product" of the vectors. You notice that the inner product of a pair of vectors is simply a number. In Chapter 1, you met the product of a row vector by a column vector: say $\begin{bmatrix} a & b \end{bmatrix}$ times $\begin{bmatrix} c \\ d \end{bmatrix}$, and found that

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac + bd \end{bmatrix},$$

the product being a 1 × 1 matrix. As you can observe, these two kinds of products are closely related; for, if V and W are the respective vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$, we have $V^t = \begin{bmatrix} a & b \end{bmatrix}$ and

$$V^tW = \begin{bmatrix} ac + bd \end{bmatrix} = \begin{bmatrix} V \bullet W \end{bmatrix}.$$

Later we shall exploit this close connection between the two products in order to deduce the algebraic properties of the inner product from the known properties of the matrix product.

Using the notion of the inner product and the formula (1) obtained above, we can state another theorem. We shall speak of the cosine of the angle included between two vectors, although we realize that we are actually referring to an angle between the associated directed line segments.

Theorem 4-5. The inner product of two vectors equals the product of the lengths of the vectors by the cosine of their included angle. Symbolically,

$$V \bullet W = ||V||\ ||W|| \cos \theta,$$

where θ is the angle between the vectors V and W.

The theorem has been proved in the case in which V and W are not collinear vectors. If we agree to take the measure of the angle between two collinear vectors to be 0° or 180° according as the vectors have the same or opposite directions, the result still holds. Indeed, as you may recall, the law of cosines on which the burden of the proof rests remains valid even when the three vertices of the "triangle" POR are collinear (Figures 4-13 and 4-14).

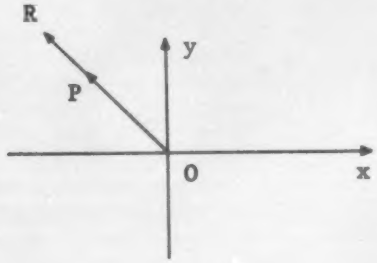


Figure 4-13. Collinear vectors in the same direction.

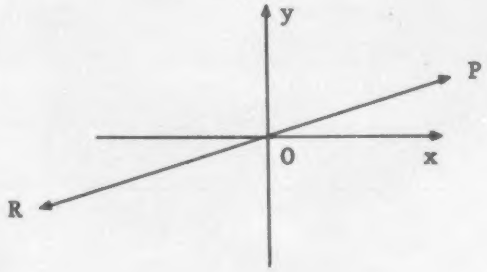


Figure 4-14. Collinear vectors in opposite directions.

Corollary 4-5-1. The relationship

$$V \bullet V = ||V||^2$$

holds for every vector V.

The corollary follows at once from Theorem 4-5 by taking $V = W$, in which case $\theta = 0^\circ$. To be sure, the result also follows immediately from the facts that, for any vector $V = \begin{bmatrix} a \\ b \end{bmatrix}$, we have

$$V \bullet V = a^2 + b^2, \text{ while } ||V|| = \sqrt{a^2 + b^2}.$$

We have examined a geometrical facet of the inner product of two vectors, but let us now look at some of its algebraic properties. Does it satisfy commutative, associative, or other algebraic laws you have met in studying number systems?

It is easy to show that the commutative law holds, that is,

$$V \bullet W = W \bullet V.$$

For if V and W are any pairs of 2×1 matrices, a computation shows that

$$V^t W = W^t V,$$

But

$$V^t W = \begin{bmatrix} V \bullet W \end{bmatrix}, \text{ while } W^t V = \begin{bmatrix} W \bullet V \end{bmatrix}.$$

Hence

$$V \bullet W = W \bullet V.$$

It is equally easy to show that the associative law cannot hold for inner products. Indeed, the products $V \bullet (W \bullet U)$ and $(V \bullet W) \bullet U$ are meaningless. To evaluate $V \bullet (W \bullet U)$, for examples, you are asked to find the inner product of the vector V with the number $W \bullet U$. But the inner product is defined for two row vectors or two column vectors and not for a vector and a number. Incidentally, you are cautioned not to confuse the product $V(W \bullet U)$ with the meaningless $V \bullet (W \bullet U)$. The former product has meaning, for it is the product of the vector V by the number $W \bullet U$.

In the exercises that follow, you will be asked to consider some of the other possible properties of the inner product. In particular, you will be asked to prove the following theorem, the first part of which was proved above.

Theorem 4-6. If V, W, and U are column vectors of order 2, and r is a real number, then

(a) $V \bullet W = W \bullet V$,

(b) $(rV) \bullet W = r(V \bullet W)$,

(c) $V \bullet (W + U) = V \bullet W + V \bullet U$,

(d) $V \bullet V \geq 0$; and

(e) if $V \bullet V = 0$, then $V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Exercises 4-5

1. Compute the cosines of the angle determined by the arrows representing the two vectors in each of the following pairs:

(a) $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$; (e) $\begin{bmatrix} -6 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 12 \end{bmatrix}$;

(b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; (f) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$;

(c) $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$; (g) $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$;

(d) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$; (h) $\begin{bmatrix} 2t \\ t \end{bmatrix}$, $\begin{bmatrix} -t \\ 2t \end{bmatrix}$.

In which cases, if any, are the arrows perpendicular? In which cases, if any, are the arrows collinear?

2. Let

$$E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Show that, for every nonzero vector V,

$$\frac{V \bullet E_1}{\|V\|} \quad \text{and} \quad \frac{V \bullet E_2}{\|V\|}$$

are the direction cosines of V.

3. (a) Prove that two vectors V and W are collinear if and only if

$$V \bullet W = \pm \|V\| \; \|W\|.$$

Explain the significance of the sign of the right-hand side of this equation.

(b) Prove that

$$(V \bullet W)^2 \leq ||V||^2 \; ||V||^2$$

and write this inequality in terms of the entries of V and W.

(c) Show also that $V \bullet W \leq ||V|| \; ||W||$.

4. Two vectors V and W are said to be orthogonal if the arrows representing V and W are perpendicular to each other. The zero vector is said to be orthogonal to every vector. Prove that V and W are orthogonal if and only if

$$V \bullet W = 0.$$

5. Fill in the blanks in the following statements so as to make the resulting sentences true:

(a) The vectors $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -10 \\ \underline{\quad} \end{bmatrix}$ are collinear.

(b) The vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -6 \\ \underline{\quad} \end{bmatrix}$ are orthogonal.

(c) The vectors $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ are ________________.

(d) The vectors $\begin{bmatrix} -18 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} \underline{\quad} \\ 12 \end{bmatrix}$ are collinear.

(e) For every positive real number t, the vectors

$$\begin{bmatrix} 3t \\ 2t \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ \underline{\quad} \end{bmatrix} \text{ are orthogonal.}$$

(f) For every negative real number t, the vectors

$$\begin{bmatrix} 3t \\ 2t \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ \underline{\quad} \end{bmatrix} \text{ are orthogonal.}$$

6. Verify that parts (a) – (d) of Theorem 4–6 are true if

$$V = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad W = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad V = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad \text{and} \quad r = 4.$$

7. Prove Theorem 4-6

   (a) by using the definition of the inner product of two vectors;

   (b) by using the fact that the matrix product V^tW satisfies the equation

$$V^tW = V \bullet W .$$

8. Prove that $||V + W||^2 = (V + W) \bullet (V + W) = ||V||^2 + 2\, V \bullet W + ||W||^2$ for every pair of vectors V and W.

9. Show that, in each of the following sets of vectors, V and W are orthogonal, V and T are collinear, and T and W are orthogonal:

   (a) $V = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad T = \begin{bmatrix} -4 \\ 8 \end{bmatrix}, \quad W = \begin{bmatrix} -6 \\ -3 \end{bmatrix};$

   (b) $V = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad T = \begin{bmatrix} 14 \\ 21 \end{bmatrix}, \quad W = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$

   Do the same relationships hold for the set

$$V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad T = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad W = \begin{bmatrix} 2 \\ 7 \end{bmatrix} ?$$

10. Let V be a nonzero vector. Suppose that W and V are orthogonal, while T and V are collinear. Show that W and T are then orthogonal.

11. Show that, for every set of real numbers r, s, and t, the vectors

$$\begin{bmatrix} r \\ s \end{bmatrix} \quad \text{and} \quad t \begin{bmatrix} -s \\ r \end{bmatrix} \quad \text{are orthogonal.}$$

12. Let $V = \begin{bmatrix} u \\ v \end{bmatrix}$, where V is not the zero vector. Show that if W and V are orthogonal, there exists a real number t such that

$$W = t \begin{bmatrix} -v \\ u \end{bmatrix} .$$

13. Show that the vectors V and W are orthogonal if and only if

$$||V + W||^2 - ||V - W||^2 = 0.$$

14. Show that if $A = \begin{bmatrix} a \\ b \end{bmatrix}$ and $B = \begin{bmatrix} c \\ d \end{bmatrix}$, then

$$||A||^2 \; ||B||^2 - (A \bullet B)^2 = (ad - bc)^2.$$

15. Show that the vectors V and W are orthogonal if and only if

$$(V + W) \bullet (V + W) = V \bullet V + W \bullet W.$$

16. Show that the equation

$$(V + W) \bullet (V - W) = V \bullet V - W \bullet W$$

holds for all vectors V and W.

17. Show that the inequality

$$||V + W|| \leq ||V|| + ||W||$$

holds for all vectors V and W.

4-6. An Area and a Determinant

Before leaving the basic properties of our geometrical interpretation for vectors, let us look at one more bit of geometry. In Section 4-4, we saw that two noncollinear vectors determine a parallelogram. That is, if

$$A = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c \\ d \end{bmatrix}$$

are two noncollinear vectors, then the points P(a, b), 0:(0,0), R:(c,d) and S:(a + c, b + d) are the vertices of a parallelogram (Figure 4-15.) A reasonable question to ask is: How can we determine the area of the parallelogram PORS?"

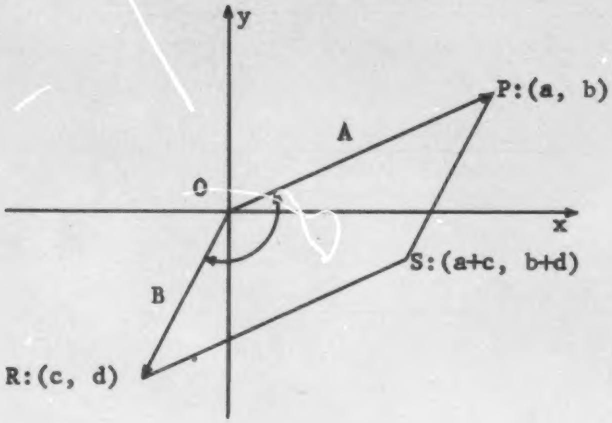


Figure 4-15. A parallelogram determined by vectors.

As you recall, the area of a parallelogram equals the product of the lengths of its base and its altitude. Thus, in Figure 4-16, the area of the parallelogram KLMN is b_1h, where b_1 is the length of side NM and h is the length of the altitude KD.

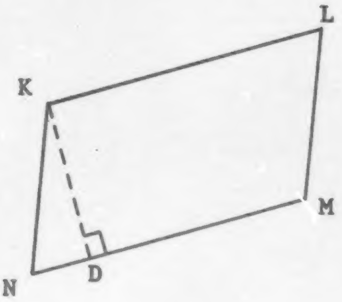


Figure 4-16. Determination of the area of a parallelogram.

But if b_2 is the length of side NK, and θ is the measure of either angle NKL or angle KNM, we have

$$h = b_2 |\sin \theta|.$$

Hence, the area of the parallelogram equals $b_1b_2 |\sin \theta|$.

Returning to Figure 4-15 and letting θ be the angle between the vectors A and B, we can now say that if G is the area of parallelogram PORS, then

$$G^2 = ||A||^2\ ||B||^2 \sin^2 \theta.$$

Now

$$\sin^2 \theta = 1 - \cos^2 \theta.$$

It follows from Theorem 4–5 that

$$\cos^2 \theta = \frac{(A \bullet B)^2}{||A||^2\ ||B||^2};$$

therefore,

$$\sin^2 \theta = \frac{||A||^2\ ||B||^2 - (A \bullet B)^2}{||A||^2\ ||B||^2}.$$

Thus, we have

$$G^2 = ||A||^2\ ||B||^2 - (A \bullet B)^2.$$

It follows from the result of Exercise 14 of the preceding section that

$$G^2 = (ad - bc)^2.$$

Therefore,

$$GK = |ad - bc|.$$

But $ad - bc$ is the value of the determinant $\delta(D)$, where D is the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$. For easy reference, let us write our result in the form of a theorem.

Theorem 4–6. The area of the parallelogram determined by the vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ equals $|\delta(D)|$, where $D = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

Corollary 4-6-1. The vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ are collinear if and only if $\delta(D) = 0$.

The argument proving the corollary is left as an exercise for the reader.

You notice that we have been led to the determinant of a 2×2 matrix in examining a geometrical interpretation of vectors. The role of matrices in this interpretation will be further investigated in Chapter 5.

Exercises 4-6

1. Let $\overrightarrow{OP}$ represent the vector A, and $\overrightarrow{OT}$ the vector B. Determine the area of triangle TOP if

   (a) $A = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$;

   (b) $A = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$;

   (c) $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

2. Compute the area of the triangle with vertices:

   (a) (0,0), (1,3), and (-3,1);

   (b) (0,0), (5,2), and (-10,-4);

   (c) (1,0), (0,1), and (2,3);

   (d) (1,1), (2,2), and (0,5);

   (e) (1,2), (-1,3), and (1,0).

4-7. The Interplay between Algebra and Geometry; Vector Analysis

In this chapter, we have developed a geometrical representation—namely, directed line segments — for 2×1 matrices, or column vectors. Guided by the definition of the algebraic operation of addition of vectors, we have found the "parallelogram law of addition" of directed line segments. The multiplication of a vector by a number has been represented by the expansion or contraction of the corresponding directed line segment by a factor equal to the number, with the sign of the factor determining whether or not the direction of the line segment is reversed. Thus, from a set of algebraic elements we have produced a set of geometric elements. Geometrical observations in turn led us back to additional algebraic concepts.

This interplay between algebra and geometry, however, is not merely an interesting intellectual exercise. The mathematics of directed line segments to which our algebra has led us forms the beginnings of a discipline called "vector analysis," which is an important tool in classical and modern physics, as well as in geometry. The "free" vectors that you meet in physics and use to represent forces, velocities, and other concepts, are close relatives of our geometric vectors, which are bound to the origin. The study in which we are engaged, consequently, is of vital importance for physicists, engineers, and other applied scientists, as well as for mathematicians.

Chapter 5

TRANSFORMATIONS OF THE PLANE

5-1. Vector Spaces and Subspaces

You have discovered that one of the most fundamental concepts in your study of mathematics is the notion of function. In geometry, the function concept appears in the idea of transformation. It is the aim of this chapter to recall what we mean by a function, to define geometric transformation, and to explore the role of matrices in the study of a significant class of these transformations.

Let us use the symbol H for the set of all real column vectors of order 2. Thus, if R is the set of real numbers, we have

$$H = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \mid u \in R \text{ and } v \in R \right\}.$$

The set H together with the operations of addition of vectors and of multiplication of a vector by a real number is an example of a type of algebraic system called a vector space.

Definition 5-1. Any set of elements is a vector space over the set of real numbers provided the following conditions are satisfied:

The sum of any two elements of the set is also an element of the set.

The product of any element of the set by a real number is also an element of the set.

The laws I and II of Theorem 4-1 hold.

A simple example of a vector space over the real numbers is the set of all linear and constant polynomials with real coefficients, that is, the set

$$\left\{p \mid p(x) = ax + b, \quad a \in R \text{ and } b \in R\right\},$$

where the addition is the usual addition of polynomials.

Another vector space over R is the set of vectors collinear with $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, that is, the set

$$\left\{r \begin{bmatrix} 2 \\ 3 \end{bmatrix} \mid r \in R\right\}.$$

This vector space is contained in H. It is called a subspace of H in accordance with the following definition.

Definition 5–2. Any nonempty subset F of H is a subspace of H provided (a) the sum of every pair of vectors of F is in F, and (b) each product of a vector in F with a real number is in F.

You may wonder what subsets of H are subspaces. First of all, to be a subspace, a given subset F must contain at least one vector, say V. Furthermore, F must also contain each of the products rV for real numbers r; that is, if V is not the zero vector then F must contain every vector collinear with V. In particular, the zero vector,

$$0V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

must belong to F. Consequently, the following theorem is true:

Theorem 5–1. Each subspace F of H contains all vectors collinear with any nonzero vector in F. In particular, each subspace contains the zero vector.

It is easy to see that the set consisting only of the zero vector is a subspace. It is also simple to verify that the set of all vectors collinear with any given nonzero vector is a subspace. With a little more effort, you can show that subsets of these two types are the only subspaces of H, other than H itself.

Theorem 5–2. Every subspace of H consists of exactly one of the following: the zero vector; the set of vectors collinear with a given nonzero vector; the space H itself.

Proof. If F is a subspace containing only one vector, then

$$F = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} ,$$

since the zero vector belongs to every subspace.

If F contains a nonzero vector V, then F contains all the vectors rV for real r. Accordingly, if all vectors of F are collinear with V, it follows that

$$F = \left\{ rV \mid r \in R \right\} .$$

But if F contains a vector W not collinear with V, we shall now prove that F is actually equal to H.

Let the noncollinear vectors V and W in the subspace F be represented by the noncollinear position vectors $\overrightarrow{OP}$ and $\overrightarrow{OR}$, respectively. Let Z be any vector of H, and let Z be represented by $\overrightarrow{OT}$. Since $\overrightarrow{OP}$ and $\overrightarrow{OR}$ are not collinear, any line parallel to one of them must intersect the line containing the other. Draw the lines through T parallel to $\overrightarrow{OP}$ and $\overrightarrow{OR}$, and let S and Q be the points in which these lines intersect the lines containing $\overrightarrow{OR}$ and $\overrightarrow{OP}$, respectively; see Figure 5–1. Then

$$\overrightarrow{OT} = \overrightarrow{OQ} + \overrightarrow{OS}.$$

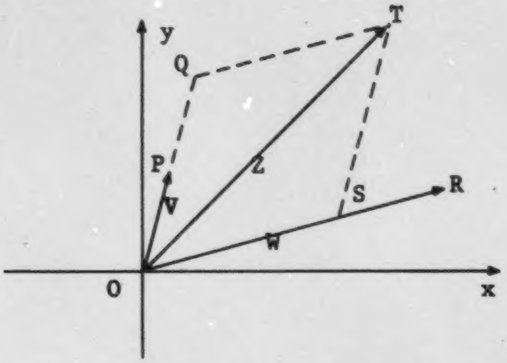


Figure 5-1. Representation of an arbitrary vector Z as a linear combination of a given pair of noncollinear vectors V and W.

But $\overrightarrow{OQ}$ is collinear with $\overrightarrow{OP}$ and $\overrightarrow{OS}$ with $\overrightarrow{OR}$. Therefore, real numbers a and b exist such that

$$\overrightarrow{OQ} = a\overrightarrow{OP} \text{ and } \overrightarrow{OS} = b\overrightarrow{OR}.$$

Hence,

$$Z = aV + bW. \tag{1}$$

Since F is a subspace, it contains aV, bW, and their sum, Z. Thus, every vector Z of H must belong to F; that is, H is a subset of F. But F is given to be a subset of H. Accordingly, F = H.

Equation (1) could have been derived by a purely algebraic argument. You will be asked to give that argument below, in Exercise 5-1-9.

Definition 5-3. If a vector Z can be expressed in the form aV + bW, where a and b are real numbers and V and W are vectors, then Z is called a linear combination of V and W.

Thus, by Definitions 5-2 and 5-3, we have the following result:

Theorem 5–3. A subspace F contains every linear combination of each pair of vectors in F.

Further, in proving Theorem 5–2, we have incidentally established the useful fact stated in the following theorem:

Theorem 5–4. Each vector of H can be expressed as a linear combination of each pair of noncollinear vectors in H.

For example, to express

$$Z = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

as a linear combination of

$$V = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} -3 \\ 4 \end{bmatrix},$$

we must determine real numbers a and b such that

$$\begin{aligned} \begin{bmatrix} 5 \\ 10 \end{bmatrix} &= a \begin{bmatrix} 4 \\ 3 \end{bmatrix} + b \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 4a - 3b \\ 3a + 4b \end{bmatrix}. \end{aligned}$$

Thus, we must solve the set of equations

$$\begin{aligned} 5 &= 4a - 3b, \\ 10 &= 3a + 4b. \end{aligned}$$

We readily find the unique solution $a = 2$ and $b = 1$; that is, we have

$$Z = 2V + W.$$

If you observe that the given vectors V and W in the foregoing example are orthogonal, that is, $V \bullet W = 0$ (see Exercise 4-5-4 on page 183), then a second method of solution might occur to you. For, if

$$Z = aV + bW,$$

then for the products $Z \bullet V$ and $Z \bullet W$ you have

$$Z \bullet V = a\,||V||^2 \quad \text{and} \quad Z \bullet W = b\,||W||^2.$$

But

$$Z \bullet V = 50, \quad Z \bullet W = 25, \quad ||V||^2 = 25, \text{ and } \quad ||W||^2 = 25.$$

Hence,

$$50 = 25a \quad \text{and} \quad 25 = 25b;$$

thus,

$$a = 2 \quad \text{and} \quad b = 1.$$

It is worth noting that the representation of a vector Z as a linear combination of two given noncollinear vectors is unique; that is, if the vectors V and W are not collinear, then for each vector Z the coefficients a and b can be chosen in exactly one way (Exercise 5-1-14, below) so that

$$Z = aV + bW.$$

The pair of noncollinear vectors V and W is called a basis for H, while the ordered pair of real numbers, a and b, are called the coordinates of Z relative to that basis. In the example above, we found that the vector $\begin{bmatrix} 5 \\ 10 \end{bmatrix}$ has coordinates 2 and 1 relative to the basis $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

Exercises 5-1

1. Express each of the following vectors as linear combinations of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, and illustrate your answers graphically:

   (a) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$,

   (b) $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$,

   (c) $\begin{bmatrix} 0 \\ -3 \end{bmatrix}$,

   (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

   (e) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

   (f) $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$,

   (g) $\begin{bmatrix} -8 \\ -6 \end{bmatrix}$,

   (h) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

   (i) $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

2. Determine the coordinates of each of the vectors in parts (a) through (i) of Exercise 1 relative to the basis $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

3. Express the vector $\begin{bmatrix} u \\ v \end{bmatrix}$ as a linear combination of the basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; this basis is called the natural basis for H.

4. Prove that the following set is a subspace of H:

$$\left\{ r \begin{bmatrix} 2 \\ 3 \end{bmatrix} \mid r \in R \right\}.$$

5. Prove that, for any given vector W, the set $\{rW \mid r \in R\}$ is a subspace of H.

6. Prove that the set of polynomials $ax^2 + bx + c$, for real numbers a, b, and c, is a vector space over the real numbers. Find two distinct subspaces of this vector space.

7. For

$$V = \begin{bmatrix} u \\ v \end{bmatrix},$$

   determine which of the following subsets of H are subspaces:

   (a) all V with $u = 0$,

   (b) all V with v equal to an integer,

   (d) all V with $2u - v = 0$,

   (e) all V with $u + v = 2$,

(c) all V with u rational, (f) all V with $uv = 0$.

8. Prove that F is a subspace of H if and only if F contains every linear combination of two vectors in F.

9. Give a purely algebraic proof of Theorem 5–2.

10. Show that $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ cannot be expressed as a linear combination of the vectors $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -15 \end{bmatrix}$.

11. Describe the set of all linear combinations of two given collinear vectors.

12. Let F_1 and F_2 be subspaces of H. Prove that the set F of all vectors belonging to both F_1 and F_2 is also a subspace.

13. In proving Theorem 5–2, we showed that if V and W are not collinear vectors, then each vector of H can be expressed as a linear combination of V and W. Prove the converse: If each vector of H has a representation as a linear combination of V and W, then V and W are not collinear.

14. Prove that if V and W are not collinear, then the representation of any vector Z in the form $aV + bW$ is unique; that is, the coefficients a and b can be chosen in exactly one way.

15. Use Equation (1) on page 142 to show that any vector

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

can be expressed uniquely as a linear combination of the basis vectors

$$\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}.$$

5-2. Functions and Geometric Transformations

You recall that a function from a set A to a set B is a correspondence between the elements of the two sets such that with each element of A there is associated exactly one element of B. The set A is the domain of the function and the set B is the range of the function. In your previous work, the functions you met generally had sets of real numbers both for domain and for range. Thus the function symbolized in the form

$$x \longrightarrow x^2$$

is likely to be interpreted as associating the real number x^2 with the nonnegative real number x. Here you have a simple example of a "real function" of a "real variable."

In Chapter 4, however, you met a function $V \longrightarrow ||V||$ having for its domain the vector space H, and for its range the set of nonnegative real numbers. In the present chapter, we shall consider functions that have their range as well as their domain in H. Specifically, we want to find a geometric interpretation for these "vector functions" of a "vector variable"; this is a continuation of the discussion started on page 127.

Such a vector function will associate, with the point P having coordinates (x,y), a point P' with coordinates (x',y'). In more vivid geometric language, we would say that the function maps the point P onto the point P'. Or we may say that it maps the geometric vector $\overrightarrow{OP}$ onto the geometric vector $\overrightarrow{OP'}$. The function can, therefore, be viewed as a process for "transforming" or mapping the plane into itself; that is to say, it is a process that associates with each point P of the plane some point P' of this plane. We shall call this process a transformation of the plane into itself or a geometric transformation. As a matter of fact, these transformations are often called "point transformations" in contrast to more general mappings in which a point may be carried into a line, a circle, or some other geometric configuration. For us, then, a geometric transformation is a helpful means of visualizing a vector function of a vector variable. As a matter of convenient terminology, we shall call the vector that such a function associates with a given vector V, the image of V; furthermore, we shall say that the function maps V onto its image.

Let us look at the simple function

$$V \longrightarrow 2V, \quad V \in H.$$

This function maps each vector V onto the vector that has the same direction as V, but that is twice as long as V. Another way of asserting this is to say that the function associates with each point P of the plane a point P' such that P and P' lie on the same ray through the origin, but $||\overrightarrow{OP'}|| = 2||\overrightarrow{OP}||$; see Figure 5–2. You may therefore think of the function in this example as uniformly stretching the plane by a factor 2 in all directions from the origin. (Under this mapping what is the point onto which the origin is mapped?)

As a second example, consider the function

$$V \longrightarrow -V, \quad V \in H.$$

This time, each vector is mapped onto the vector having length equal and direction opposite to that of the given vector. Viewed as a point transformation, the function associates with any point P its "reflection" in the origin; see Figure 5–3.

The function

$$V \longrightarrow -2V$$

combines both of the effects of the preceding functions, so that the vector associated with V is twice as long as V, but has the opposite direction to that of V.

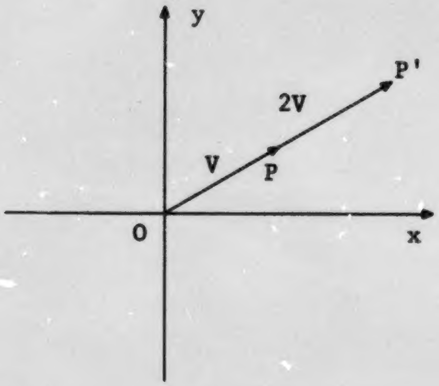


Figure 5–2. The transformation $V \longrightarrow 2V$.

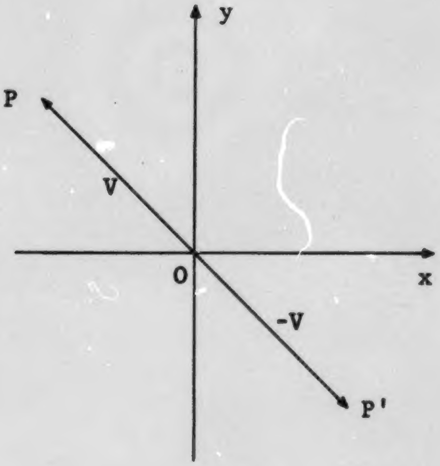


Figure 5–3. The transformation $V \longrightarrow -V$.

Now, let us look at the function

$$V \longrightarrow ||V||\ V.$$

As in our first example, each vector is mapped by the function onto a vector having the same direction as the given vector. Indeed, every vector of length 1 is its own image. But if $||V|| > 1$, then the image of V has a length greater than that of V, with the expansion factor increasing with the length of V itself. Thus, the vector

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

having length 2, is mapped onto

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix},$$

which is twice as long. The vector

$$\begin{bmatrix} 5 \\ 12 \end{bmatrix},$$

whose length is 13, has the image

$$\begin{bmatrix} 65 \\ 156 \end{bmatrix},$$

with length 169. On the other hand, for nonzero vectors of length less than one, we obtain image vectors of shorter length, the contraction factor decreasing with decreasing length of the original vector. Thus,

$$\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} \text{ is mapped onto } \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix},$$

the image being half as long as the given vector. Again, the vector

$$\begin{bmatrix} -\frac{4}{7} \\ -\frac{3}{7} \end{bmatrix} \text{ is mapped onto } \begin{bmatrix} -\frac{20}{49} \\ -\frac{15}{49} \end{bmatrix},$$

the length of the first vector being 5/7, while the length of its image is only $(5/7)^2$, or 25/49. Although we may try to think of this mapping as a kind of stretching of the plane in all directions from the origin, so that any point and its image are collinear with the origin, this mental picture has also to take into account the fact that the degree of expansion varies with the distance of a given point from the origin, and that for points within the circle of radius 1 about the origin the so-called stretching is actually a compression.

The mapping

$$V \longrightarrow \frac{1}{2}(V + U), \quad \text{where } U = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

can be written in the form

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} \frac{x+2}{2} \\ \frac{y+1}{2} \end{bmatrix} .$$

Therefore, if you remember the formulas for the coordinates of a linear segment in terms of the coordinates of the endpoints of the segment, you recognize that this function maps each point P onto the midpoint of the line segment joining P to the point (2,1). One way of visualizing this mapping is to regard it as displacing or translating the plane in the direction of the vector U through a distance equal to the length of U and then compressing the plane by the factor 1/2; see Figure 5-4.

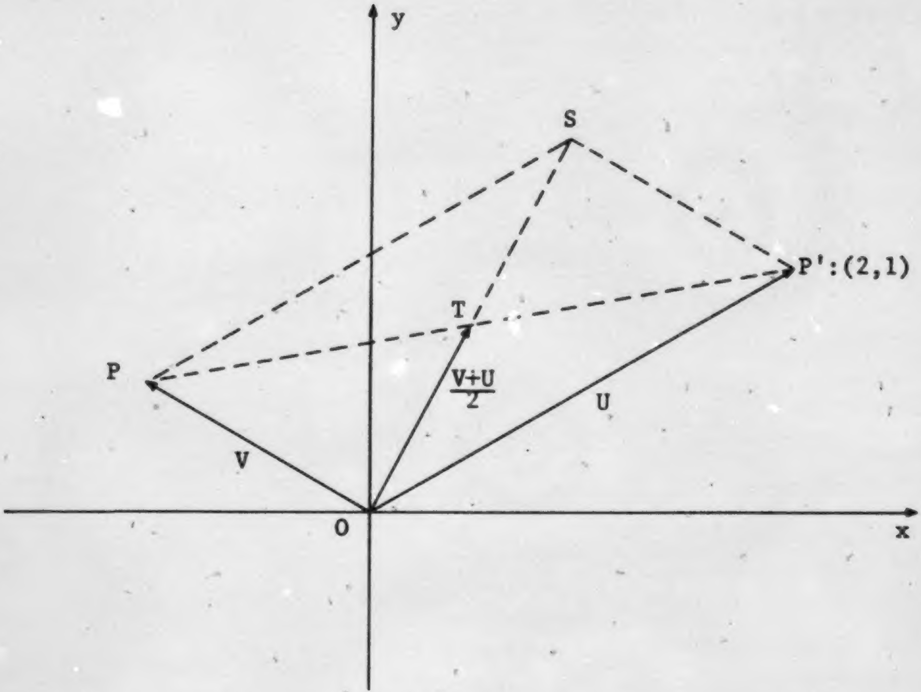


Figure 5-4. The transformation $V \longrightarrow \frac{V+U}{2}$, where $U = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Still another example of a vector function on H is the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x + 2y \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} .$$

Under this mapping, each point is moved parallel to the x axis through a distance equal to twice the ordinate of the point. The result is a horizontal shearing of the plane (Figure 5–5), with points above the x axis being moved to the right and points below that axis moved to the left.

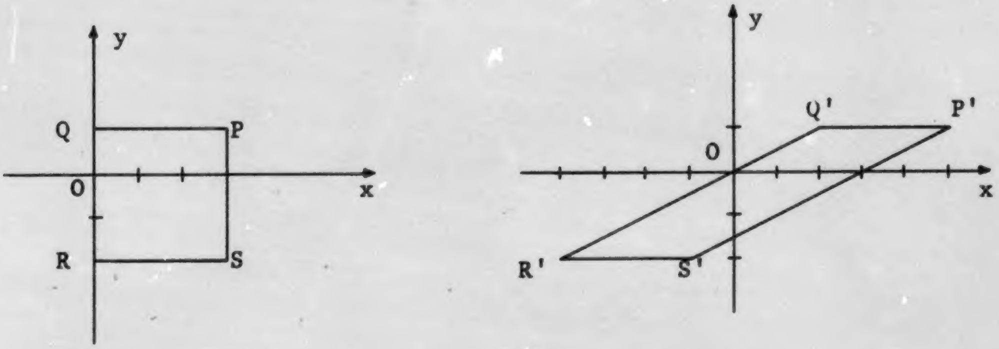


Figure 5–5. The transformation $\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x + 2y \\ y \end{bmatrix}$.

All the vector functions discussed above map distinct points of the plane onto distinct points. But we can certainly produce functions not having this property. Thus, the function

$$V \longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

maps every point of the plane onto the origin.

On the other hand, the transformation

$$V = \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ 0 \end{bmatrix} \tag{2}$$

maps the point (x, y) onto the point of the x axis that has the same first component as V. For example, every point of the line $x = 3$ is mapped onto the point (3,0). Since the image of each point P can be located by drawing a perpendicular line from P to the x axis, we may think of P as being carried or projected on the x axis by a line perpendicular to this axis. Consequently, this mapping may be described as a perpendicular or othogonal projection of the plane on the x axis. You notice that these last two functions (1) and (2) map H onto subspaces of H.

Since we have met examples of transformations that map distinct points onto distinct points and have also seen transformations under which distinct points may have the same image, it is useful to define a new term to distinguish between these two kinds of vector functions.

Definition 5-4. A transformation from the set H onto the set H is one-to-one provided the images of distinct vectors are also distinct vectors.

Thus, if f is a function from H to H and if we write f(V) for the image of V under the transformation f, then Definition 5-4 can be formulated symbolically as follows: The function f is a one-to-one transformation on H provided

$$V \neq U$$

implies

$$f(V) \neq f(U)$$

for vectors V and U in H.

Exercises 5-2

1. Find the image of the vector V under the mapping

$$V \longrightarrow 3V$$

for each of the following values of V:

(a) $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$,

(b) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$,

(c) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

(d) $\begin{bmatrix} 7 \\ -3 \end{bmatrix}$,

(e) $\begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ -3 \end{bmatrix}$,

(f) $\begin{bmatrix} 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 7 \\ -3 \end{bmatrix}$.

2. Find $f(V)$ under the mapping

$$f : V = \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} y \\ 0 \end{bmatrix}$$

for each of the following values of V:

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

(b) $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$,

(c) $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$,

(d) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

(e) $5 \begin{bmatrix} -1 \\ -3 \end{bmatrix}$,

(f) $-2 \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

3. Describe the geometric effect of each of the following transformations of H on the vector $V = \begin{bmatrix} x \\ y \end{bmatrix}$:

(a) $V \longrightarrow V$,

(b) $V \longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

(c) $V \longrightarrow aV,\ a > 0$,

(d) $V \longrightarrow -aV,\ a > 0$,

(e) $V \longrightarrow \begin{bmatrix} 0 \\ y \end{bmatrix}$,

(f) $V \longrightarrow \begin{bmatrix} y \\ y \end{bmatrix}$,

(g) $V \longrightarrow \begin{bmatrix} -x \\ y \end{bmatrix}$,

(h) $V \longrightarrow \begin{bmatrix} x \\ -y \end{bmatrix}$,

(i) $V \longrightarrow \begin{bmatrix} 2x \\ y \end{bmatrix}$,

(j) $V \longrightarrow \begin{bmatrix} 3x \\ 3y \end{bmatrix}$,

(k) $V \longrightarrow \begin{bmatrix} x + y \\ y \end{bmatrix}$,

(l) $V \longrightarrow \begin{bmatrix} x \\ 2x + y \end{bmatrix}$,

(m) $V \longrightarrow \begin{bmatrix} x - 2y \\ y \end{bmatrix}$,

(n) $V \longrightarrow \begin{bmatrix} x \\ y - 3x \end{bmatrix}$.

4. Determine which of the transformations in the preceding exercise are one-to-one.

5. Find expressions of the type $V \longrightarrow V'$ for the transformations of H that map each point P onto the point P' related to P in the ways described below:

(a) P' is one unit to the right of P and four units above P;

(b) P' is the perpendicular projection of P on the horizontal line through (3,2);

(c) P' is the perpendicular projection of P on the vertical line through (−1,−2);

(d) $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are collinear but opposite in direction, and $||\overrightarrow{OP'}|| = \frac{1}{2}||\overrightarrow{OP}||$;

(e) P' is the intersection of the horizontal line through P with the line of slope −1 passing through the origin (horizontal projection on the line $y = -x$);

(f) P' is the intersection of the vertical line through P with the line $y = 2x$ (vertical projection on the line $y = 2x$).

6. Show that the mapping of H into itself that sends each point P into the point of intersection of the line $y = x$ with the line through P having slope 2 is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} 2x - y \\ 2x - y \end{bmatrix}.$$

7. (a) Show that the mapping

$$V = \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x + 2y \\ 4x + 3y \end{bmatrix}$$

can be expressed in the form

$$V \longrightarrow \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} V.$$

(b) Find the image under this transformation of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) Find the image under this transformation of the subspace of vectors collinear with $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

8. Solve parts (b) and (c) of Exercise 7 when $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is replaced by

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$,

(b) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$, (d) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

9. Under the transformation given in Exercise 7, find by two different methods the image of each of the following vectors:

(a) $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$,

(b) $\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, (e) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$,

(c) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, (f) $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

10. Consider the mapping

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(a) Find the images under this mapping of the pair of points (5,1) and (1,−2), and show that the distance between the given pair of points equals the distance between their images.

(b) Solve part (a) if the given points are (−2,10) and (6,−5).

(c) Solve part (a) if the given points are (a,b) and (c,d).

5–3. Matrix Transformations

As noted earlier, especially in Chapter 3, the pair of equations

$$a_{11}x + a_{12}y = b_1,$$
$$a_{21}x + a_{22}y = b_2,$$

can be written in the form

$$AV = B,$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad V = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Consequently, in solving the equations you actually determine all the vectors V that are mapped onto the particular vector B by the function

$$V \longrightarrow AV. \qquad (1)$$

The study of the solution of systems of linear equations thus leads to the consideration of the special class of transformations on H that are expressible in the form (1), where A is any 2×2 matrix with real entries. These matrix transformations constitute a very important class of mappings, having extensive applications in mathematics, statistics, physics, operations research, and engineering.

An important property of matrix transformations is that they are linear mappings; that is, they preserve vector sums and the products of vectors with real numbers.

Let us formulate these ideas explicitly.

Definition 5–5. A linear transformation on H is a function f from H into H such that

(a) for every pair of vectors V and U in H, we have

$$f(V + U) = f(V) + f(U);$$

(b) for every real number r and every vector V in H, we have

$$f(rV) = r\,f(V).$$

Theorem 5–5. Every matrix transformation is linear.

Proof. Let f be the transformation

$$f : V \longrightarrow AV,$$

where A is any real matrix of order 2. We must show that for any vectors V and U, we have

$$A(V + U) = AV + AU;$$

further, we must show that for any vector V and any real number r we have

$$A(rV) = r(AV).$$

But these equalities hold in virtue of parts III (a) and III (f) of Theorem 4-2 (see page 160).

The linearity property of matrix transformations can be used to derive the following result concerning transformations of the subspaces of H.

Theorem 5-6. A matrix A maps every subspace F of H onto a subspace F' of H.

Proof. Let F' denote the set of vectors

$$\{AU \mid U \in F\}.$$

To prove that F' is a subspace of H, we must show that the following statements are true:

(a) For any pair of vectors P', Q' in F', the sum $P' + Q'$ is in F'.

(b) For any vector P' in F' and any real number r, rP' is in F'.

If P' and Q' are in F', then they must be the images of vectors P and Q in F; that is,

$$P' = AP,$$
$$Q' = AQ.$$

It follows that

$$P' + Q' = AP + AQ = A(P + Q),$$

and $P' + Q'$ is the image of the vector $P + Q$ in F. (Can you tell why $P + Q$ is in F?) Hence, $(P' + Q') \in F'$. Similarly,

$$rP' = r(AP) = A(rP),$$

and hence rP' is the image of rP. But $rP \in F$ because F is a subspace. Thus, rP' is the image of a vector in F; therefore, $rP' \in F'$.

Corollary 5-6-1. Every matrix maps the plane H onto a subspace, either the origin, or a straight line through the origin, or H itself.

For example, to determine the subspaces onto which

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

maps

(a) $F = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; y = -3x \right\}$,

(b) H itself,

we proceed as follows.

For (a), the vectors of F are of the form

$$U = \begin{bmatrix} x \\ -3x \end{bmatrix} = x \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad x \in R.$$

Hence,

$$AU = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \cdot x \begin{bmatrix} 1 \\ -3 \end{bmatrix} = x \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = x \begin{bmatrix} -2 \\ -1 \end{bmatrix} = -x \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Thus, F is mapped onto F', the set of vectors collinear with $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$; that is,

$$F' = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; y = \tfrac{1}{2} x \right\}.$$

In other words, A maps the line passing through the origin with slope -3 onto the line through the origin with slope 1/2.

As regards (b), we note that for any vector

$$V = \begin{bmatrix} x \\ y \end{bmatrix} \in H,$$

we have

$$AV = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x + 2y \\ 2x + y \end{bmatrix} = (2x + y) \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Since $2x + y$ assumes all real values as x and y run over the set of real numbers, it follows that H is also mapped onto F'; that is, A maps the entire plane onto the line

$$y = \frac{1}{2} x.$$

Exercises 5.3

1. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. For each of the following values of the vector V,

   (a) $V = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

   (b) $V = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$,

   (c) $V = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$,

   (d) $V = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$,

   (e) $V = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$,

   (f) $V = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$,

   determine:

   (i) the vector into which A maps V,

   (ii) the line onto which A maps the line containing V.

2. A certain matrix maps

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ into } \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ into } \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Using this information, determine the vector into which the matrix maps each of the following:

(a) $\begin{bmatrix}3\\3\end{bmatrix}$ $\left(\text{Hint: } \begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}\right)$.

(b) $\begin{bmatrix}4\\2\end{bmatrix}$, (e) $\begin{bmatrix}1\\1\end{bmatrix}$,

(c) $\begin{bmatrix}1\\-1\end{bmatrix}$, (f) $\begin{bmatrix}4\\5\end{bmatrix}$,

(d) $\begin{bmatrix}-2\\2\end{bmatrix}$, (g) $\begin{bmatrix}2\\3\end{bmatrix}$.

3. Consider the following subspaces of H:

$$F_1 = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}, \qquad F_2 = \left\{\begin{bmatrix}x\\y\end{bmatrix} \,\middle|\, y = 2x\right\},$$

$$F_3 = \left\{\begin{bmatrix}x\\y\end{bmatrix} \,\middle|\, y = -2x\right\}, \quad F_4 = H \text{ itself.}$$

Determine the subspaces onto which F_1, F_2, F_3, and F_4 are mapped by each of the following matrices:

(a) $A = \begin{bmatrix}-2 & 1\\-2 & 1\end{bmatrix}$, (b) $B = \begin{bmatrix}0 & 1\\-2 & 3\end{bmatrix}$, (c) AB, (d) BA.

4. Let $A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$.

(a) Calculate AV for

$$V = \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}p\\q\end{bmatrix}.$$

(b) Find the vector V for which

$$AV = \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}r\\s\end{bmatrix}.$$

5. Determine which of the following transformations of H are linear, and justify your answer:

(a) $V = \begin{bmatrix}x\\y\end{bmatrix} \longrightarrow \begin{bmatrix}x+1\\y\end{bmatrix}$, (d) $V \longrightarrow \begin{bmatrix}2x\\5y\end{bmatrix}$,

(b) $V \longrightarrow \begin{bmatrix} x \\ 1 \end{bmatrix}$, (e) $V \longrightarrow \frac{1}{2}(V + U)$, where $U = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

(c) $V \longrightarrow \begin{bmatrix} x - y \\ x + y \end{bmatrix}$, (f) $V \longrightarrow \|V\|\, V$.

6. Show that the matrix A maps the plane onto the origin if and only if

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

7. Show that the matrix A maps every vector of the plane onto itself if and only if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

8. Show that

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

maps the line $y = 0$ onto itself. Is any point of that line mapped onto itself by this matrix?

9. (a) Show that each of the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

maps H onto the x axis.

(b) Determine the set of all matrices that map H onto the x axis. (Hint: You must determine all possible matrices A such that corresponding to each $V \in H$ there is a real number r for which

$$AV = r \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \qquad (1)$$

In particular, (1) must hold for suitable r when V is replaced by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.)

10. Determine the set of all matrices that map H onto the y axis.

11. (a) Determine the matrix A such that

$$AV = 2V$$

for all V.

(b) The mapping

$$V \longrightarrow aV \qquad (a > 0)$$

multiplies the lengths of all vectors without changing their directions. It amounts to a change of scale. The number a is accordingly called a scale factor or scalar. Find the matrix A that yields only a change of scale:

$$AV = aV.$$

12. Prove that for every matrix A the set F of all vectors U for which

$$AU = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is a subspace of H. This subspace is called the kernel of the mapping.

13. (a) Show that the matrix of a transformation is determined when the images of 2 noncollinear vectors are given.

(b) Find the matrix that maps

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ onto } \begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ onto } \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

14. Prove that if a linear transformation of H maps each of 2 noncollinear vectors onto itself, then the transformation maps every vector onto itself; that is, the transformation is the identity mapping.

15. Prove that a transformation f of H into itself is linear if and only if

$$f(rV + sU) = r\, f(V) + s\, f(U)$$

for every pair of vectors V and U of H and every pair of real numbers r and s.

In the preceding section, we proved that every matrix represents a linear transformation of H into H. We now prove the converse: Every linear transformation of H into H can be represented by a matrix.

Theorem 5–7. Let f be a linear transformation of H into H. Then, relative to any given basis for H, there exists one and only one matrix A such that, for all $V \in H$,

$$AV = f(V).$$

Proof. We prove first that there cannot be more than one matrix representing f. Suppose that there are two matrices A and B such that, for all $V \in H$,

$$AV = f(V) \quad \text{and} \quad BV = f(V).$$

Then

$$AV - BV = f(V) - f(V)$$

for each V. Hence,

$$(A - B)V = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ for all } V \in H.$$

Thus, $A - B$ maps every vector onto the origin. It follows (Exercise 5–3–6) that $A - B$ is the zero matrix; therefore,

$$A = B.$$

Hence, there is at most one matrix representation of f.

Next, we show how to find the matrix representation for the linear transformation f. Let S_1 and S_2 be a pair of noncollinear vectors of H. Let

$$f(S_1) = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad \text{and} \quad f(S_2) = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

be the respective images of S_1 and S_2 under the mapping f. If V is any vector of H, it follows from Theorem 5-4 that there exist real numbers v_1 and v_2 such that $V = v_1S_1 + v_2S_2$. Since f is a linear transformation, we have

$$f(V) = f(v_1S_1 + v_2S_2) = v_1f(S_1) + v_2f(S_2).$$

Accordingly,

$$f(V) = v_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix}.$$

Thus,

$$f(V) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

It follows that f is represented by the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

when vectors are expressed in terms of their coordinates relative to the basis S_1, S_2.

You notice that the matrix A is completely determined by the effect of f on the pair of noncollinear vectors used as the basis for H. Thus, once you know that a given transformation on H is linear, you have a matrix representing the mapping when you have the images of the natural basis vectors,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For example, it can be shown by a geometric argument that the counterclockwise rotation of the plane through an angle of 30° about the origin is a linear transformation. This function maps any point P onto the point P', where the measure of the angle POP' is equal to 30° (Figure 5-6). It is easy to see (Figure 5-7) that

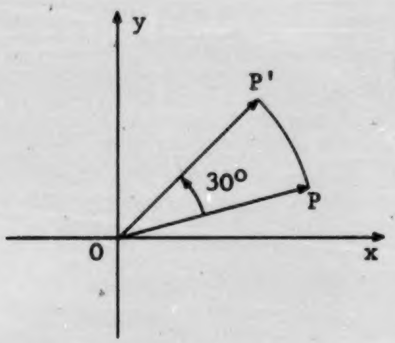


Figure 5-6. A rotation through an angle of 30° about the origin.

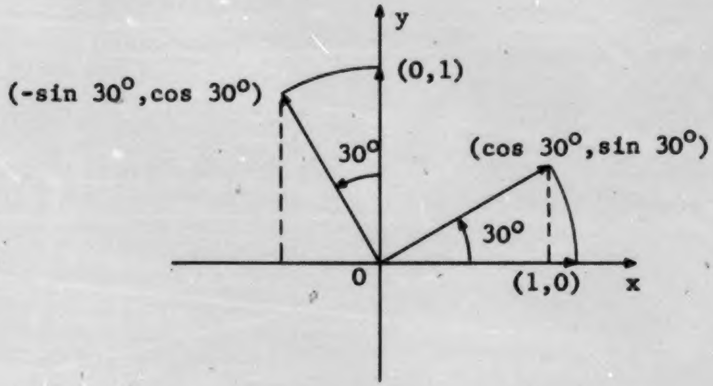


Figure 5-7. The images of the points (1,0) and (0,1) under a rotation of 30° about the origin.

$$\begin{bmatrix}1\\0\end{bmatrix} \text{ is mapped onto } \begin{bmatrix}\cos 30^\circ\\ \sin 30^\circ\end{bmatrix}$$

and

$$\begin{bmatrix}0\\1\end{bmatrix} \text{ is mapped onto } \begin{bmatrix}-\sin 30^\circ\\ \cos 30^\circ\end{bmatrix} .$$

Thus, the matrix representing this rotation is

$$A = \begin{bmatrix}\cos 30^\circ & -\sin 30^\circ\\ \sin 30^\circ & \cos 30^\circ\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix} .$$

Note that the first column of A is the vector onto which $\begin{bmatrix}1\\0\end{bmatrix}$ is mapped;

the second column of A is the image of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The product or composition of two transformations is defined just as you define the composition of two real functions of a real variable.

Definition 5-6. If f and g are transformations on H, then for each vector V in H the composition transformations fg and gf are the transformations such that

$$fg(V) = f(g(V)) \quad \text{and} \quad gf(V) = g(f(V)).$$

Thus, to find the image of V under the transformation fg, you first apply g, and then apply f. Consequently, if g maps V onto U, and if f maps U onto W, then fg maps V onto W.

The following theorem is readily proved (Exercise 5-4-7).

Theorem 5-8. If f is a linear transformation represented by the matrix A, and g is a linear transformation represented by the matrix B, then fg and gf are both linear transformations; fg is represented by AB, while gf is represented by BA.

For example, suppose that in the coordinate plane each position vector is first reflected in the vertical axis, and then the resulting vector is doubled in length. Let us find a matrix representation of the resulting linear transformation on H. If g is the mapping that transforms each vector into its reflection in the vertical axis, then we have

$$g : \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

If f maps each vector into twice the vector, then we have

$$f : \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow 2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Accordingly, the matrix representing fg is

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Exercises 5-4

1. Show that each of the mappings in Exercise 5-2-3 is linear, by determining matrices representing the mappings.

2. Consider the linear transformations,

    p: reflection in the horizontal axis,

    q: horizontal projection on the line $y = -x$ (Exercise 5-2-5e),

    r: rotation counterclockwise through 90°,

    s: shear moving each point vertically through a distance equal to the abscissa of the point,

    of H into H. In each of the following, determine the matrix representing the given transformation:

    (a) p,
    (b) q,
    (c) r,
    (d) s,
    (e) pq,
    (f) qp,
    (g) pr,
    (h) rp,
    (i) qs,
    (j) sq,
    (k) $s(rs)$,
    (l) $(sr)s$,
    (m) $p(sq)$,
    (n) $(ps)q$,
    (o) $(sp)(rq)$.

3. Let f be the rotation of the plane counterclockwise through 45° about the origin, and let g be the rotation clockwise through 30°. Determine a matrix representing the rotation through 15° about the origin.

4. (a) Show that every linear transformation maps the origin onto itself.

(b) Show that every linear transformation maps every subspace of H onto a subspace of H.

5. For every two linear transformations f and g on H, define f + g to be the transformation such that, for each $V \in H$,

$$(f + g)(V) = f(V) + g(V).$$

Without using matrices, prove that f + g is a linear transformation on H.

6. For each linear transformation f on H and each real number a, define af to be the transformation such that

$$af(V) = f(aV).$$

Without using matrices, prove that af is a linear transformation on H.

7. Prove Theorem 5-8.

8. Without using matrices, prove each of the following:

(a) $f(g + h) = fg + fh$,

(b) $(f + g)h = fh + gh$,

(c) $f(ag) = a(fg)$,

where f, g, and h are any linear transformations on H and a is any real number.

5-5. One-to-one Linear Transformations

The reflection of the plane in the x axis clearly maps distinct points onto distinct points; thus, the reflection is a one-to-one linear transformation on H. Moreover, the reflection maps any pair of noncollinear vectors onto a pair of noncollinear vectors. It is easy to show that this property is common to all one-to-one linear transformations of H into itself.

Theorem 5–9. Every one-to-one linear transformation on H maps noncollinear vectors onto noncollinear vectors.

Proof. Let S_1 and S_2 be a pair of noncollinear vectors and let

$$f(S_1) = T_1 \quad \text{and} \quad f(S_2) = T_2$$

be their images under the one-to-one linear mapping f. Since f is one-to-one, we know that T_1 and T_2 are not both the zero vector. We may suppose, therefore, that T_1 is not the zero vector. To show that T_1 and T_2 are not collinear, we shall demonstrate that the assumption that they are collinear leads to a contradiction.

If T_1 and T_2 are collinear, then there exists a real number r such that $T_2 = r\,T_1$. Now, consider the image under f of the vector $r\,S_1$. Since f is linear, we have

$$\begin{aligned} f(r\,S_1) &= r\,f(S_1) \\ &= r\,T_1 \\ &= T_2. \end{aligned}$$

Thus, each of the vectors $r\,S_1$ and S_2 is mapped onto T_2. Since f is one-to-one, it follows that

$$r\,S_1 = S_2,$$

and therefore that S_1 and S_2 are collinear vectors. But this contradicts the fact that S_1 and S_2 are not collinear. Hence, the assumption that T_1 and T_2 are collinear must be false. Consequently, f must map noncollinear vectors onto noncollinear vectors.

Corollary 5–9–1. The subspace onto which a one-to-one linear transformation maps H is H itself.

Proof. Since the subspace contains a pair of noncollinear vectors, the corollary follows by use of Theorems 5–3 and 5–4.

The link between one–to–one transformations on H and second–order matrices having inverses is given in the next theorem.

Theorem 5–10. Let f be a linear transformation represented by the matrix A. Then f is one–to–one if and only if A has an inverse.

Proof. Suppose that A has an inverse. Let S_1 and S_2 be vectors in H having the same image under f. Now,

$$f(S_1) = AS_1 \quad \text{and} \quad f(S_2) = AS_2.$$

Thus,

$$AS_1 = AS_2.$$

Hence,

$$\begin{aligned} A^{-1}(AS_1) &= A^{-1}(AS_2), \\ (A^{-1}A)S_1 &= (A^{-1}A)S_2, \\ IS_1 &= IS_2, \end{aligned}$$

and

$$S_1 = S_2.$$

Thus, f must be a one–to–one transformation.

On the other hand, suppose that f is one–to–one. From Theorem 5–9, it follows that every vector in H is the image of some vector in H. In particular, there are vectors W and U such that

$$f(W) = AW = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$f(U) = AU = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Accordingly, the matrix having for its first column the vector W, and for its second column the vector U, is the inverse of A.

Corollary 5–10–1. A linear transformation represented by the matrix A is one–to–one if and only if

$$\delta(A) \neq 0.$$

The theory of systems of two linear equations in two variables can now be studied geometrically. Writing the system

$$\begin{aligned} a_{11}x + a_{12}y &= u, \\ a_{21}x + a_{22}y &= v, \end{aligned} \tag{1}$$

in the form

$$AV = U, \tag{2}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad V = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} u \\ v \end{bmatrix},$$

we seek the vectors V that are mapped by the matrix A onto the vector U.

If $\delta(A) \neq 0$, we now know that A represents a one–to–one mapping of H onto H. Therefore, A maps exactly one vector V onto U, namely, $V = A^{-1}U$. Thus, the system (1) — or, equivalently, (2) — has exactly one solution.

If $\delta(A) = 0$, then, in virtue of Corollary 4–6–1, the columns of A must be collinear vectors. (Hence, A must have one of the forms

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} a & ra \\ b & rb \end{bmatrix},$$

where not both a and b are zero. If A has the first of these forms, then A maps H onto the origin. In the other two cases, A maps H onto the line of vectors collinear with the vector $\begin{bmatrix} a \\ b \end{bmatrix}$. (See Exercise 5-5-7, below.) With these results in mind, you may now complete the discussion of the solution of Equation (2).

Exercises 5-5

1. Using Theorem 5-10 or its corollary, determine which of the transformations in Exercise 5-2-3 are one-to-one.

2. Show that a linear transformation is one-to-one if and only if the kernel of the mapping consists only of the zero vector. (See Exercise 5-3-12.)

3. (a) Show that if f is a one-to-one linear transformation on H, then there exists a linear transformation g such that, for all $V \in H$,

$$gf(V) = V$$

and

$$fg(V) = V.$$

The transformation g is called the inverse of f and is usually written $g = f^{-1}$.

(b) Show that the transformation $g = f^{-1}$ in part (a) is a one-to-one transformation on H.

4. Prove that the set of one-to-one linear transformations on H is a group relative to the operation of composition of transformations.

5. Prove that if f and g are one-to-one linear transformations of H, then fg is also a one-to-one transformation of H.

6. Show that if f and g are linear transformations of H such that fg is a one-to-one transformation, then both f and g are one-to-one transformations.

7. (a) Show that if $\delta(A) = 0$, then the matrix A maps H onto a point (the origin) or onto a line.

(b) Show that if A is the zero matrix and U is the zero vector, then every vector V of H is a solution of the equation $AV = U$.

(c) Show that if $\delta(A) = 0$, but A is not the zero matrix, then the solution set of the equation

$$AV = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is a set of collinear vectors.

(d) Show that if $\delta(A) = 0$, but A is not the zero matrix and U is not the zero vector, then the solution set of the equation

$$AV = U$$

either is empty or consists of all vectors of the form

$$\{V_1 + tV_2 \mid t \in R\},$$

where V_1 and V_2 are fixed vectors such that

$$AV_1 = U \text{ and } AV_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

8. Show that if the equation $AV = U$ has more than one solution for any given U, then A does not have an inverse.

5–6. Invariant Subspaces

The reflection in the x axis,

$$f : V \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} V,$$

evidently has the property of mapping each vector (point) on the x axis onto itself. If you think of a mapping as "carrying" a vector onto its image, you might think of the vectors on the x axis as being held fixed in this reflection. The notion of fixed vectors or points is important enough for us to formalize the idea in a definition.

Definition 5–7. If a transformation of H into itself maps a given vector onto itself, then that vector is a fixed vector for the transformation. A fixed vector is also called an invariant vector.

Reflection in the x axis leaves fixed no points other than those on this axis. However, it is easy to see that each point on the y axis is mapped by this transformation onto another point of the y axis, except for the origin. If W is any vector on the y axis, $W \neq \underline{0}$, then

$$f(W) = -W.$$

Thus, the vectors collinear with W form a fixed, or invariant, subspace of H for this transformation.

Definition 5–8. A subspace F of H is an invariant subspace for a given transformation provided: (a) the image of every vector in F is also a vector in F, and (b) every vector in F is the image of some vector in F.

The following theorem shows the connection between invariant vectors and invariant subspaces under linear transformations.

Theorem 5–11. If W is an invariant vector for a linear transformation f, then every vector in the subspace $F = \{rW \mid r \in R\}$ is invariant under the transformation; that is, F is an invariant subspace.

Proof. Since f is a linear transformation that maps W onto itself, we have

$$\begin{aligned} f(rW) &= rf(W) \\ &= rW. \end{aligned}$$

Thus, f maps rW onto itself for every real value of r.

To determine the invariant subspaces of a linear transformation f, let us suppose that f is represented by the matrix A. We seek vectors W and real numbers c such that

$$AW = cW.$$

If I is the identity matrix of order 2, we then have

$$AW = (cI)W,$$

that is,

$$AW - (cI)W = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$(A - cI)W = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (1)$$

Letting

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } W = \begin{bmatrix} x \\ y \end{bmatrix},$$

we may rewrite equation (1) as follows:

$$\begin{bmatrix} a_{11} - c & a_{12} \\ a_{21} & a_{22} - c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (2)$$

We know there is a nonzero vector W satisfying equations (2) if and only if

$$\delta(A - cI) = 0,$$

that is,

$$(a_{11} - c)(a_{22} - c) - a_{12}a_{21} = 0,$$

or

$$c^2 - (a_{11} + a_{22})c + \delta(A) = 0. \qquad (3)$$

Equation (3) is called the *characteristic equation* of the matrix A and its roots are called the *characteristic values* or *roots* of A. Once this quadratic equation is solved for c, the corresponding vectors W satisfying equation (2) are readily found (Exercise 5–5–7c).

You should notice that invariant vectors of A correspond to a characteristic root equal to 1.

For example, to determine the invariant subspaces of the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix},$$

we must solve the matrix equation

$$\begin{bmatrix} 2 - c & 3 \\ 0 & 1 - c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For the characteristic equation, we obtain

$$c^2 - 3c + 2 = 0,$$

the roots of which are $c = 1$ and $c = 2$.

For $c = 1$, equation (2) becomes

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This matrix equation is equivalent to the system

$$x + 3y = 0,$$
$$0x + 0y = 0.$$

Thus, A maps the line $x = -3y$ onto itself; that is, the subspace of vectors collinear with $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is invariant. Actually, since $c = 1$, each vector of this subspace is invariant.

For $c = 2$, equation (2) becomes

$$\begin{bmatrix} 0 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$3y = 0,$$
$$-1y = 0.$$

Hence, A maps the line $y = 0$ onto itself; that is, the invariant subspace corresponding to $c = 2$ is the set of vectors collinear with $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. But in this subspace, only $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an invariant vector.

Definition 5-9. Each nonzero vector W satisfying the equation

$$AW = cW$$

is called a characteristic vector corresponding to the characteristic value c of A.

The determination of the characteristic roots and vectors of a matrix is of vast importance in many engineering and scientific problems. The analysis of flutter and vibration phenomena, the stability analysis of an airplane, and many other physical problems require finding the characteristic roots and vectors of matrices.

Exercises 5–6

1. Determine the characteristic roots and vectors of each of the following matrices:

(a) $\begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$,

(b) $\begin{bmatrix} -3 & 4 \\ -1 & 2 \end{bmatrix}$, (d) $\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$.

2. Prove that zero is a characteristic root of a matrix A if and only if $\delta(A) = 0$.

3. Show that a linear transformation f is one-to-one if and only if zero is not a characteristic root of the matrix representing f.

4. Prove that if zero is a characteristic root of the matrix A, then A has at most one invariant subspace other than the subspace consisting of the zero vector alone. What is the maximum number of noncollinear characteristic vectors that A can have?

5. Determine the invariant subspaces (fixed lines) of the mapping given by

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}.$$

Show that these lines are mutually perpendicular.

6. The characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

in the illustrative example on page 229 is

$$c^2 - 3c + 2 = 0.$$

For matrices, the corresponding equation is

$$C^2 - 3C + 2I = \underline{0},$$

where I is the identity matrix of order 2 and $\underline{0}$ is the zero matrix of order 2. Show that A is a solution of this matrix equation; that is, show that

$$A^2 - 3A + 2I = \underline{0}.$$

7. Show that the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a solution of its characteristic (matrix) equation; that is, show that

$$A^2 - (a_{11} + a_{22})A + \delta(A)I = \underline{0}.$$

This result is the case $n = 2$ of a famous theorem called the Cayley-Hamilton Theorem, which states that an analogous result holds for matrices of any order n.

8. Show that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an invariant vector of the transformation

$$V \longrightarrow \|V\|\, V,$$

but that $2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not invariant under this mapping. Does this result contradict Theorem 5-11?

9. Show that A maps every line through the origin onto itself if and only if

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

for $r \neq 0$.

10. Let $d = (a_{11} - a_{22})^2 + 4\,a_{12}a_{21}$, where a_{11}, a_{12}, a_{21}, and a_{22} are any real numbers. Show that the number of distinct real characteristic roots of the matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is

$$\begin{array}{l} 0 \text{ if } d < 0, \\ 1 \text{ if } d = 0, \\ 2 \text{ if } d > 0. \end{array}$$

11. Find a nonzero matrix that leaves no line through the origin fixed.

12. Determine a one-to-one linear transformation that maps exactly one line through the origin onto itself.

13. Show that every matrix of the form $\begin{bmatrix} r & s \\ s & t \end{bmatrix}$ has two distinct characteristic roots if $s \neq 0$.

14. Show that the matrix A and its transpose A^t have the same characteristic roots.

5-7. Rotations and Reflections

Since length is an important property in Euclidean geometry, we shall look for the linear transformations of the plane that leave unchanged the length $||V||$ of every vector V. Examples of such transformations are the following:

(a) the reflection of the plane in the x axis,

(b) a rotation of the plane through any given angle about the origin,

(c) a reflection is the x axis followed by a rotation about the origin.

Actually, we can show that any linear transformation that preserves the lengths of all vectors is equivalent to one of these three. The following theorem will be very useful in proving that result.

Theorem 5–12. A linear transformation of H that leaves unchanged the length of every vector also leaves unchanged (a) the inner product of every pair of vectors and (b) the magnitude of the angle between every pair of vectors.

Proof. Let V and U be a pair of vectors in H and let V' and U' be their respective images under the transformation. In virtue of Exercise 4–5–8, we have

$$\|V + U\|^2 = \|V\|^2 + 2V \bullet U + \|U\|^2 \qquad (1)$$

and

$$\|V' + U'\|^2 = \|V'\|^2 + 2V' \bullet U' + \|U'\|^2. \qquad (2)$$

Since the transformation is linear, for the image of $V + U$ we have

$$(V + U)' = V' + U'.$$

Consequently, (2) can be written as

$$\|(V + U)'\|^2 = \|V'\|^2 + 2V' \bullet U' + \|U'\|^2. \qquad (3)$$

But the transformation preserves the length of each vector; thus, we obtain

$$\|V'\| = \|V\|, \quad \|U'\| = \|U\|, \quad \text{and} \quad \|(V + U)'\| = \|V + V\|.$$

Making these substitutions in equation (3), we get

$$\|V + U\|^2 = \|V\|^2 + 2V' \bullet U' + \|U\|^2. \qquad (4)$$

Comparing equations (1) and (4), you see that we must have

$$V \bullet U = V' \bullet U';$$

that is, the transformation preserves the inner product.

Since the magnitude of the angle between V and U can be expressed in terms of inner products (Theorem 4-5), it follows that the transformation also preserves that magnitude.

Corollary 5-12-1. If a linear transformation preserves the length of every vector, then it maps orthogonal vectors onto orthogonal vectors.

See Exercise 4-5-4 on page 183 for the definition of orthogonal vectors. This simply means that the geometric vectors are mutually perpendicular.

It is very easy to show the transformations we are considering also preserve the distance between every pair of points in the plane. We state this property formally in the next theorem, the proof of which is left as an exercise.

Theorem 5-13. A linear transformation that preserves the length of every vector leaves unchanged the distance between every pair of points in the plane; that is, if V' and U' are the respective images of the vectors V and U, then

$$||V' - U'|| = ||V - U||.$$

Let us now find a matrix representing any given linear length-preserving transformation of H. All we need to find are the images of the vectors

$$S_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

under such a transformation. (Why is this so?)

If S_1' and S_2' are the respective images of S_1 and S_2, then we know that both S_1' and S_2' are of length 1 and that they are orthogonal to each other.

Suppose that S_1' forms the angle α (alpha) with the positive half of the x axis (Figure 5-8). Since the length of S_1' equals 1, we have

$$S_1' = \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix}.$$

We know that S_2' is perpendicular to S_1'. Hence, there are two opposite possibilities for the direction of S_2', because the angle β (beta) that S_2' makes with the positive half of the x axis may be either

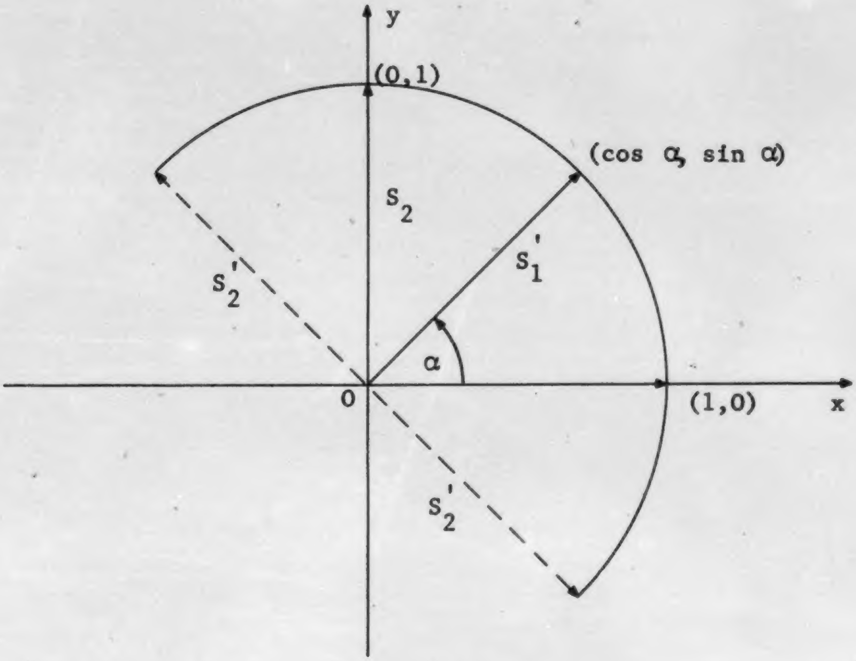


Figure 5-8. A length-preserving transformation.

$$\beta = \alpha + \frac{\pi}{2} \tag{5}$$

or

$$\beta = \alpha - \frac{\pi}{2}. \tag{6}$$

In the first case (5), we have

$$S_2' = \begin{bmatrix} \cos \beta \\ \sin \beta \end{bmatrix} = \begin{bmatrix} \cos & \left(\alpha + \frac{\pi}{2}\right) \\ \sin & \left(\alpha + \frac{\pi}{2}\right) \end{bmatrix} = \begin{bmatrix} -\sin \alpha \\ \cos \alpha \end{bmatrix}.$$

In the second case (6), we have

$$S_2' = \begin{bmatrix} \cos & \left(\alpha - \frac{\pi}{2}\right) \\ \sin & \left(\alpha - \frac{\pi}{2}\right) \end{bmatrix} = \begin{bmatrix} \sin \alpha \\ -\cos \alpha \end{bmatrix}.$$

Accordingly, any linear transformation f that leaves the length of each vector unchanged must be represented by a matrix having either the form

$$A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \tag{7}$$

or the form

$$B = \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{bmatrix}. \tag{8}$$

In the first instance (7), the transformation f simply rotates the basis vectors S_1 and S_2 through an angle α and we suspect that f is a rotation of the entire plane H through that angle. To verify this observation, we write the vector V in terms of its angle of inclination θ (theta) to the x axis and the length $r = ||V||$; that is, we write

$$V = \begin{bmatrix} r & \cos \theta \\ r & \sin \theta \end{bmatrix}. \tag{9}$$

Forming AV from equations (7) and (9), we obtain

$$AV = \begin{bmatrix} r(\cos \theta \cos \alpha - \sin \theta \sin \alpha) \\ r(\sin \theta \cos \alpha + \cos \theta \sin \alpha) \end{bmatrix}.$$

From the formulas of trigonometry,

$$\cos(\theta + \alpha) = \cos\theta \cos\alpha - \sin\theta \sin\alpha,$$

$$\sin(\theta + \alpha) = \sin\theta \cos\alpha + \cos\theta \sin\alpha,$$

we see that

$$AV = \begin{bmatrix} r\cos(\theta + \alpha) \\ r\sin(\theta + \alpha) \end{bmatrix}.$$

Thus, AV is the vector of length r at an angle $\theta + \alpha$ to the horizontal axis. We have proved that the matrix A represents a rotation of H through the angle α.

But suppose f is represented by the matrix B in Equation (8) above. This transformation differs from the one represented by A in that the vector S_2' is reflected across the line of the vector S_1'. Consequently, you may suspect that this transformation amounts to a reflection of the plane in the x axis followed by a rotation through the angle α. Since you know that the reflection in the x axis is represented by the matrix

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

you may, therefore, expect that

$$B = AJ. \tag{10}$$

We leave this verification as an exercise.

Exercises 5-7

1. Obtain the matrices that rotate H through the following angles:

(a) 180°,	(f) 90°,
(b) 45°,	(g) -120°,
(c) 30°,	(h) 360°,

(d) 60°, (i) −135°,

(e) 270°, (j) 150°.

2. Write out the matrices that represent the transformation consisting of a reflection in the x axis followed by the rotations of Exercise 1.

3. Verify Equation (10), above.

4. A linear transformation of H that preserves the length of every vector is called an orthogonal transformation, and the matrix representing the transformation is called an orthogonal matrix. Prove that the transpose of an orthogonal matrix is orthogonal.

5. Show that the inverse of an orthogonal matrix is an orthogonal matrix.

6. Show that the product of two orthogonal matrices is orthogonal.

7. (a) Show that a translation of H in the direction of the vector

$$U = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

and through a distance equal to the length of U is given by the mapping

$$V \longrightarrow V + U.$$

(b) Show that this mapping does not preserve the length of every vector, but that it does preserve the distance between every pair of points in the plane.

(c) Determine whether or not this mapping is linear.

8. Let R_α and R_β denote rotations of H through the angles α and β, respectively. Prove that a rotation through α followed by a rotation through β amounts to a rotation through $\alpha + \beta$; that is, show that

$$R_\beta R_\alpha = R_{\alpha+\beta} .$$

9. Note that the matrix A of Equation (7) is a representation of a complex

number. What does the result of Exercise 8 imply for complex numbers?

10. (a) Find a matrix that represents a reflection across the line of the vector

$$\begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix}.$$

(b) Show that the matrix B of Equation (8), above, represents a reflection across the line of some vector.

Appendix

RESEARCH EXERCISES

The exercises in this Appendix are essentially "research-type" problems designed to exhibit aspects of theory and practice in matrix algebra that could not be included in the text. They are especially suited as individual assignments for those students who are prospective majors in the theoretical and practical aspects of the scientific disciplines, and for students who would like to test their mathematical powers; or students might join forces in working them.

1. Quaternions. The algebraic system that is explored in this exercise was invented by the Irish mathematician and physicist, William Rowan Hamilton, who published his first paper on the subject in 1835. It was not until 1858 that Arthur Cayley, an English mathematician and lawyer, published the first paper on matrices. Since Hamilton's system of quaternions is actually an algebra of matrices, it is more easily presented in this guise than in the form in which it was first developed.

In this exercise, we shall consider the algebra of 2×2 matrices with complex numbers as entries. The definitions of addition, multiplication, and inversion remain the same. We use C for the set of all complex numbers and we denote by K the set of all matrices

$$\begin{bmatrix} z & w \\ z_1 & w_1 \end{bmatrix},$$

where z, w, z_1, and w_1 are elements of C. As is the case with matrices having real entries, the element

$$\begin{bmatrix} z & w \\ z_1 & w_1 \end{bmatrix}$$

of K has an inverse if and only if

$$zw_1 - wz_1 \neq 0,$$

and we have

$$\begin{bmatrix} z & w \\ z_1 & w_1 \end{bmatrix}^{-1} = \frac{1}{zw_1 - z_1w} \begin{bmatrix} w_1 & -w \\ z_1 & z \end{bmatrix}, \quad zw_1 - z_1w \neq 0.$$

Since 1 is a complex number, the unit matrix is still

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If

$$z = x + iy,$$

then we write

$$\bar{z} = x - iy$$

and call this number the complex conjugate of z, or simply the conjugate of z.

A quaternion is an element q of K of the particular form

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}, \quad z \in C \text{ and } w \in C.$$

We denote by Q the set of all quaternions.

(a) Show that $\delta(q) = x^2 + y^2 + u^2 + v^2$ if $z = x + iy$ and $w = u + iv$. Hence conclude that, since x, y, u, and v are real numbers, $\delta(q) = 0$ if and only if $q = \underline{0}$.

(b) Show that if $q \in Q$ then q has an inverse if and only if $q \neq \underline{0}$

and exhibit the form of q^{-1} if it exists.

Three elements of Q are of particular importance and we give them special names:

$$U = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

$$V = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$W = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

(c) Show that if

$$q = \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix},$$

where $z = x + iy$ and $w = u + iv$, then

$$q = xI + yU + uV + vW.$$

(d) Prove the following identities involving I, U, V and W:

$$U^2 = V^2 = W^2 = -I$$

and

$$UV = W = -VU, \quad VW = U = -WV, \quad \text{and} \quad WU = V = -UW.$$

(e) Use the preceding two exercises to show that if $q \in Q$ and $r \in Q$, then $q + r$, $q - r$, and qr are all elements of Q.

The conjugate of the element

$$q = \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix}, \quad z = x + iy, \quad w = u + iv,$$

is

$$\overline{q} = \begin{bmatrix} \overline{z} & -w \\ \overline{w} & z \end{bmatrix},$$

and the norm and trace are given respectively by

$$|q| = \left[\delta(q)\right]^{1/2}$$

and

$$t(q) = 2x.$$

(f) Show that if $q \in Q$, and if q is invertible, then

$$q^{-1} = \frac{1}{|q|}\,\bar{q}\,.$$

From this conclude that if $q \in Q$, and if q^{-1} exists, then $q^{-1} \in Q$.

(g) Show that each $q \in Q$ satisfies the quadratic equation

$$q^2 - t(q)\,q + |q|^2\,I = 0.$$

(h) Show that if $q \in Q$ then

$$q\bar{q} = |q|^2\,I.$$

Note that this may be proved by using the result that if

$$q = aI + bU + cV + dW$$

then

$$\bar{q} = aI - bU - cV - dW,$$

and then using the results given in (d).

(i) Show that if $q \in Q$ and $r \in Q$, then

$$|qr| = |q|\,|r|$$

and

$$|q + r| \leq |q| + |r|.$$

The geometry of quaternions constitutes a very interesting subject. It requires the representation of a quaternion

$$q = aI + bU + cV + dW$$

as a point with coordinates (a, b, c, d) in four-dimensional space. The subset of elements,

$$Q_1 = \{q \mid q \in Q \text{ and } |q| = 1\},$$

is a group and is represented geometrically as the hypersphere with equation

$$a^2 + b^2 + c^2 + d^2 = 1.$$

2. Nonassociative Algebras

The algebra of matrices (we restrict our attention in this exercise to the set M of 2 × 2 matrices) has an associative but not a commutative multiplication. "Algebras" with nonassociative multiplication have become increasingly important in recent years—for example, in mathematical genetics. Genetics is a subdiscipline of biology and is concerned with transmission of hereditary traits. Nonassociative "algebras" are important also in the study of quantum mechanics, a subdiscipline of physics. We give first a simple example of a Lie algebra (named after the geometer Sophus Lie).

If $A \in M$ and $B \in M$, we write

$$A \circ B = AB - BA$$

and read this "A op B," "op" being an abbreviation for operation.

(a) Prove the following properties of o :

(i) $A \circ B = - B \circ A$,

(ii) $A \circ A = \underline{0}$,

(iii) $A \circ (B \circ C) + B \circ (C \circ A) + C \circ (A \circ B) = \underline{0}$,

(iv) $A \circ I = \underline{0} = I \circ A$.

(b) Give an example to show that $A \circ (B \circ C)$ and $(A \circ B) \circ C$ are different and hence that $\circ$ is not an associative operation.

Despite these strange properties, $\circ$ behaves nicely relative to ordinary matrix addition.

(c) Show that $\circ$ distributes over addition:

$$A \circ (B + C) = (A \circ B) + (A \circ C)$$

and

$$(A + B) \circ C = (A \circ C) + (B \circ C)$$

(d) Show that $\circ$ behaves nicely relative to multiplication by a number.

It will be recalled that A^{-1} is called the multiplicative inverse of A and is defined as the element B satisfying the relationships

$$AB = I = BA.$$

But it must also be recalled that this definition was motivated by the fact that

$$AI = A = IA,$$

that is, by the fact that I is a multiplicative unit.

(e) Show that there is no $\circ$ unit.

We know, from the foregoing work, that $\circ$ is neither commutative nor associative. Here is another kind of operation, called Jordan multiplication: If $A \in M$ and $B \in M$, we define

$$AjB = \frac{(AB + BA)}{2}.$$

We see at once that

$$AjB = BjA,$$

so that *Jordan multiplication* is a commutative operation; but it is *not* associative.

(f) Determine all of the properties of the operation j that you can. For example, does j distribute over addition?

3. The Algebra of Subsets

We have seen that there are interesting algebraically defined subsets of M, the set of all 2×2 matrices. One such subset, for example, is the set Z, which is isomorphic with the set of complex numbers. Much of higher mathematics is concerned with the "global structure" of "algebras," and generally this involves the consideration of subsets of the "algebras" being studies. In this exercise, we shall generally underscore letters to denote subsets of M.

If $\underline{A}$ and $\underline{B}$ are subsets of M, then

$$\underline{A} + \underline{B}$$

is the set of all elements of the form

$$A + B, \text{ where } A \in M \text{ and } B \in M.$$

In set-builder notation this may be written

$$\underline{A} + \underline{B} = \{A + B \mid A \in \underline{A} \text{ and } B \in \underline{B}\}.$$

By an *additive subset* of M is meant a subset $\underline{A} \subset M$ such that

$$\underline{A} + \underline{A} \subset \underline{A}.$$

(a) Determine which of the following are additive subsets of M:

(i) $\{\underline{0}\}$,

(ii) $\{I\}$,

(iii) M,

(iv) Z,

(v) M_1, the set of all A in M with $\delta(A) = 1$,

(vi) The set of all elements of M whose entries are nonnegative.

(b) Prove that if $\underline{A}$, $\underline{B}$, and $\underline{C}$ are subsets of M, then

(i) $\underline{A} + \underline{B} = \underline{B} + \underline{A}$,

(ii) $\underline{A} + (\underline{B} + \underline{C}) = (\underline{A} + \underline{B}) + \underline{C}$,

(iii) and if $\underline{A} \subset \underline{B}$ then $\underline{A} + \underline{C} \subset \underline{B} + \underline{C}$.

(c) Prove that if $\underline{A}$ and $\underline{B}$ are additive subsets of M, then

$$\underline{A} + \underline{B}$$

is also an additive subset of M.

Let V denote the set of all column vectors

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

with $x \in R$ and $y \in R$.

(d) Show that if v is a fixed element of V, then

$$\left\{ A \mid A \in M \text{ and } Av = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

is an additive subset of M. Notice also that if $Av = 0$ then $(-A)v = 0$.

If $\underline{A}$ and $\underline{B}$ are subsets of M, then

$$\underline{AB}$$

is the set of all

$$AB, A \in \underline{M} \text{ and } B \in \underline{M}.$$

Using the set-builder notation, we can write this in the form

$$\underline{AB} = \{AB \mid A \in \underline{A} \text{ and } B \in \underline{B}\}.$$

A subset $\underline{A}$ of M is multiplicative if

$$\underline{AA} \subset \underline{A}.$$

(e) Which of the subsets in part (a) are multiplicative?

(f) Show that if $\underline{A}$, $\underline{B}$, and $\underline{C}$ are subsets of M, then

(i) $\underline{A}(\underline{BC}) = (\underline{AB})\underline{C}$,

(ii) and if $\underline{A} \subset \underline{B}$, then $\underline{AC} \subset \underline{BC}$

(g) Give an example to two subsets $\underline{A}$ and $\underline{B}$ of M such that

$$\underline{AB} \neq \underline{BA}.$$

(h) Determine which of the following subsets are multiplicative:

(i) $\{\underline{0}, I\}$,

(ii) $\{I, -I\}$,

(iii) the set of all elements of M with negative entries,

(iv) the set of all elements of M for which the upper left-hand entry is less than 1,

(v) the set of all elements of M of the form

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix},$$

with $0 \leq x$, $0 \leq y$, and $x + y \leq 1$.

The exercises stated above are suggestions as to how this "algebra of subsets" works. There are many other results that come to mind, but we shall leave them to you to find. Here are some clues: How would you define $t\underline{A}$ if $t \in R$ and $\underline{A} \in M$? Is $(-1)\underline{A} = -\underline{A}$? Wait a minute! What does $-\underline{A}$ mean?

What does $\underline{A}^7$ mean? Does set multiplication distribute over addition, over union, over intersection? Do not expect that even your teacher knows the answer to all of these possible questions. Few people know all of them and fewer still, of those who know them, remember them. If you conjecture that something is true but the proof of it escapes you, then try to construct an example to show that it is false. If this does not work, try proving it again, and so on.

4. Analysis and Synthesis of Proofs

This is an exercise in analysis and synthesis, taking an old proof to pieces and using the pattern to make a new proof. In describing his activities, a mathematician is likely to put at the very top that of creating new results. But "result" in mathematics usually means "theorem and proof." The mathematician does not by any means limit his methods in conjecturing a new theorem: He guesses, uses analogies, draws diagrams and figures, sets up physical models, experiments, computes; no holds are barred. Once he has his conjecture firmly in mind, he is only half through, for he still must construct a proof. One way of doing this is to analyze proofs of known theorems that are somewhat like the theorem he is trying to prove and then synthesize a proof of the new theorem. Here we ask you to apply this process of analysis and synthesis of proofs to the algebra of matrices. To accomplish this, we shall introduce some new operations among matrices by analogy with the old operations.

For simplicity of computation, we shall use only 2×2 matrices.

To start with, we introduce new operations in the set of real numbers, R. If $x \in R$ and $y \in R$, we define

$$x \wedge y = \text{the } \underline{\text{smaller}} \text{ of } x \text{ and } y \quad (\text{read: "}x\ \underline{\text{cap}}\ y\text{"})$$

and

$x \vee y$ = the larger of x and y (read: "x cup y").

(a) Show that if $x \in R$, $y \in R$, and $z \in R$, then

(i) $x \wedge y = y \wedge x$,

(ii) $x \vee y = y \vee x$,

(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,

(iv) $x \vee (y \vee z) = (x \vee y) \vee z$,

(v) $x \wedge x = x$,

(vi) $x \vee x = x$,

(vii) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(viii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Although the foregoing operations may seem a little unusual, you will have no difficulty in proving the above statements. They are not difficult to remember if you notice the following facts:

The even-numbered results can be obtained from the odd-numbered results by interchanging $\wedge$ and $\vee$, and conversely.

The first states that $\wedge$ is commutative and the third states that $\wedge$ is associative. The fifth is new but the seventh states that $\wedge$ distributes over $\vee$.

To define the matrix operations, let us think of $\wedge$ as the analog of multiplication and $\vee$ as the analog of addition and let us begin with our new matrix "multiplication."

We define

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \wedge \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} (a \wedge x) \vee (b \vee z) & (a \wedge y) \vee (b \wedge w) \\ (c \wedge x) \vee (d \wedge z) & (c \wedge y) \vee (d \wedge w) \end{bmatrix}.$$

This is simply the row by column operations, except that $\wedge$ is used in place of multiplication and $\vee$ is used in place of addition. To see this more clearly, we write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix} .$$

(b) Write out a proof that if A, B, and C are elements of M, then

$$A(BC) = (AB)C.$$

Be sure not to omit any steps in the proof. Using this as a pattern, write out a proof that

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C,$$

verifying at each step that you have the necessary results from (a) to make the proof sound. List all the properties of the two pairs of operations that you need, such as associativity, commutativity, and distributivity.

(c) Using the analogy between $\vee$ and addition, define $A \vee B$ for elements A and B of M.

(d) State and prove, for the new operations, analogues of all the rules you know for the operations of matrix addition and multiplication.

BIBLIOGRAPHY

C. B. Alledoerfer and C. O. Oakley, "Fundamentals of Freshman Mathematics," McGraw-Hill Book Company, Inc., New York, 1959.

R. A. Beaumont and R. W. Ball, "Introduction to Modern Algebra and Matrix Theory," Rinehart and Company, New York, 1954.

Garrett Birkhoff and Saunders MacLane, "A Survey of Modern Algebra," The Macmillan Company, New York, 1959.

R. E. Johnson, "First Course in Abstract Algebra," Prentice-Hall, Inc., New York, 1958.

M. J. Weiss, "Higher Algebra for the Undergraduate," John Wiley and Sons, Inc., New York, 1949.

INDEX

Aaron, 2
Abelian group, 99
Abelian property of binary operation, 99
Abstract group, 98–99
Addition of matrices, 11–12, 19–20
  associativity law for, 14–15
  commutativity law for, 13
"Addition" symbol $\sum$, 35
Addition of vectors, 10
  geometric interpretation of, 171–173
Additive inverse, 15
Algebra, 112–113
  embedding of, 110
  global structure of, 247
  nonassociative, 245
  richness of, 110
  of subsets, 247
Angle between two vectors, 178
Anticommutative matrices, 55
Area and determinant function, 185–188
Associative law for addition, 14–15, 56, 58, 62
  of matrices, 14–15
Associative law for multiplication, 47, 56, 58, 62
  of matrices, 47
Associative property of binary operation, 99
Atom, 120
Backward solution, 122
Basis, 196
  coordinates of a vector relative to, 196
  natural, 197
Binary operation, 98–99
  abelian property of, 99
  associative property of, 99
  closure property of, 99
  commutative property of, 99
  identity property of, 99
  inverse property of, 99
Brackets used in designating matrices, 4
Cancellation law for multiplication, 42
  invalidity of, for matrices, 41–42
Cap operation, 250
Cartesian coordinate plane, 165
Cayley, v
Cayley–Hamilton theorem, 232
Characteristic equation, 229
Characteristic root, 229
Characteristic value, 229
Characteristic vector, 230
Clock face, group property of addition associated with, 99–100
Closure property of binary operation, 99
Collinear vectors, 166
Column matrix, 5
Column of a matrix, 3
Column vector, 5
  order of, 159–160
Combination, linear, 194
Commutative group, 99
Commutative law for addition, 13, 56, 58, 62
Commutative law for multiplication, 41, 56, 58, 62
  invalidity of, for matrices, 40–41
Commutative property of binary operation, 99
Complex numbers, 1, 10–11, 103–111, 241
  isomorphism of, with matrices, 103–111
Composition transformation, 219
Computing machines, v, 117
Conformability for addition, 11
Conformability for multiplication, 31
Conjugate, of a complex number, 242
  of a quaternion, 243
Contraction factor, 201
Coordinate plane, Cartesian, 165
Coordinates of a vector, 196
Correspondence, one-to-one, 106
Cosines, direction, 164
  law of, 177
Cup operation, 251
Determinant function, 85–86
  of multiplicative inverse, 88
  of a product, 87
  related to an area, 185–188
  of the transpose of a matrix, 92
Diagonal method, 117, 139–141
Diagonal, principal, 140
Difference of matrices, 16, 19–21
Direction cosines, 164
Distribution laws for multiplication over addition, 47–48, 56, 58, 62
Domain, 127–128, 199
Electronic computing machine, v, 117
"Element" symbol $\in$, 12
Elementary matrix, 132
  inverse of, 132–133
Elimination, method of, 116
Entry of a matrix, 3

Equality of matrices, 8
Equation, characteristic, 229
Expansion factor, 201
Factor, contraction, 201
  expansion, 201
Field, 58, 63
Finite group, 101
Fixed point, 227
Fixed vector, 227
Forward solution, 119–122
Function, 127, 199
  determinant, 85–86
  matrix, 127
Galois, 101
Geometric transformation, 191, 199
Geometric vector, 163
Global structure of algebra, 247
Group, 93
  abelian, 99
  abstract, 98–99
  commutative, 99
  finite, 101
  of integers, 100–101
  of invertible matrices, 93
  related to face of clock, 99–100
Hamilton, 232, 241
Heisenberg, v
Identity element for addition, 12
Identity element for multiplication, 39, 52
Identity map, 205
Identity property of binary operation, 99
Identity transformation, 205
Image, 200
Importance of matrices, v, 2
Inequalities for vectors, 183, 185
Infinite decimal, 1
Inner product, 177–179
  algebraic properties of, 181–182
Invariant subspace, 227
Inverse matrix function, 143
Inverse, multiplicative, 57, 71
  of elementary matrix, 132–133
  of matrix of order two, 83
  of a product, 89
  uniqueness of, 75
Inverse property of binary operation, 99
Isomorphism, 109–110
Jordan multiplication, 246
Kernel, 205
Law of cosines, 177
Left multiplication, 41
Length of a vector, 164
Length-preserving transformation, 233
Linear combination, 194
Linear equations (see Systems of linear equations)
Linear map, 209
Linear transformation, 209
Lopez, 2
Mantle, 2–3
Map, 199
  identity, 215
  linear, 209
Matrices, 3
  addition of (see Addition of matrices)
  anticommutative, 55
  conformable for addition, 11
  difference of, 16, 19–21
  equality of, 8
  identity element for addition of, 12
  importance of, v, 2
  multiplication of (see Multiplication of matrices)
  notation for, 6
  product of, 34
  subtraction of, 16, 19–21
  sum of, 11–12, 19–20
  additive inverse of, 16
  brackets used in designating, 4
  column, 5
  elementary, 132
  inverse of, 132–133
  entry of, 3
  column of, 3
  multiplication of, by a number, 22–25
  multiplicative inverse of, 57, 71
  for matrices of order two, 83
  uniqueness of, 75
  negative of, 16
  order of, 4
  orthogonal, 239
  row, 5
  row of, 3
  square, 4
  square, order of, 5
  transpose of, 6
Matrix function, 127
  inverse, 143
Matrix transformation, 208
Molecule, 120
Multiplication, Jordan, 246
Multiplication of matrices, 27–36
  associative law for, 47
  conformability for, 31
  distribution laws for, over addition, 47–48
  identity element for, 39, 52
  invalidity of cancellation law for, 41–42

Multiplication of matrices (continued),
  left, 41
  right, 41
  by zero matrix, 51
Multiplication of matrix by number, 22–25
Multiplicative inverse of elementary matrix, 132–133
Multiplicative inverse of a matrix, 57, 71
  of order two, 83
  uniqueness of, 75
Multiplicative inverse of a product, 89
Natural basis, 197
Negative of a matrix, 16
Negative of a number, 15
Negative whole number, 1
Nonassociative algebra, 245
  operation in, 245
Norm are of a quaternion, 244
  of a vector, 164
Notation for matrices, 6
Null vector, 164
Number, 1
  complex, 1
  negative whole, 1
  positive whole, 1
  real, 1
  zero, 1
One–to–one correspondence, 106
One–to–one transformation, 205
Operation in a nonassociative algebra, 245
Order of a matrix, 4
Orthogonal vectors, 183
Parallelogram law, 173, 188
Perpendicular vectors, 177
Point, fixed, 227
Polynomials, residue classes of, 110
Positive whole number, 1
Principal diagonal, 140
Product of matrices, 34
Product matrix, 30–31
Product of matrix and number, 23
Product of vector and number, geometric interpretation of, 167–168
Programming, 117
Projection, 205
  orthogonal, 205
  perpendicular, 205
Pythagorean theorem, 163
Quaternion, 241–245
  norm of, 244
  trace of, 244
Range, 127–128, 199
Real number, 1
Reciprocal of a matrix (see Inverse, multiplicative)
Reflection in a line, 219, 233
Reflection in the origin, 200
Root, characteristic, 229
Right multiplication, 41
Ring, 58, 65–68
  with an identity element, 58
Rotation, 218, 233
Row matrix, 5
Row of a matrix, 3
Row vector, 5
Scalar, 215
Set M of 2 × 2 matrices, 56
  closure of, with respect to addition, 56
  closure of, with respect to multiplication, 56
Shearing, 204
Sigma notation, 35
Slope, 164
Solution, backward, 122
Solution, forward, 119–122
Space, vector, 191
Square matrix, 4
  order of, 5
Square roots, 43–44
Structure of algebras, 247
Structure of mathematics, iii, v
Subset, additive, 247
Subsets, algebra of, 247
Subspace, 192
  invariant, 227
Subtraction of matrices, 16, 19–21
Substitution, method of, 115
Subtraction of vectors, geometric interpretation of, 175
Sum of matrices, 11–12, 19–20
Systems of linear equations, 115
  elimination method for, 116
  equivalent, 116
  in general, 151–156
  "line" of solutions of, 153
  substitution method for, 115
Trace of a quaternion, 244
Transformation, 191
  composition, 219
  geometric, 191, 199
  identity, 205
  inverse, 225
  length–preserving, 233
  linear, 209
  matrix, 208
  one–to–one, 205

Transformation (continued),
  onto, 200
  orthogonal, 239
  of the plane into itself, 199
Transformations, composition of, 219
  product of, 219
Transpose of a matrix, 6
Triangulation method, 117, 119–125
  solution by matrices, 130–138
  in terms of matrices, 125–129
Unit circle, 97
Unit matrix, 52
Value, characteristic, 229
Vector, 5
  characteristic, 230
  column, 5
  fixed, 227
  geometric, 163
  length of, 164
  multiplication by a number, geometrical interpretation of, 167–168
  norm of, 164
  null, 164
  representation of, by directed line segment, 162–163
  row, 5
  zero, 164
Vector analysis 188–189
Vector function, 199
Vector space, 112, 191
Vectors, 5
  addition of, 10
    geometric interpretation of, 171–173
    parallelogram law for, 173, 188
  angle between, 178
  bound, 189
  collinear, 166
  collinear, in opposite directions, 180
  collinear, in same direction, 180
  free, 189
  inequalities for, 183, 185
  inner product of, 177–179
  algebraic properties of, 181–182
  orthogonal, 183
  perpendicular, 177
  subtraction of, 16, 19–21
    geometric interpretation of, 175
Whole number, 1
Williams, 2
Zero, 1
Zero matrix, 12, 16
  as identity matrix for addition, 12
  multiplication by, 51
Zero vector, 164